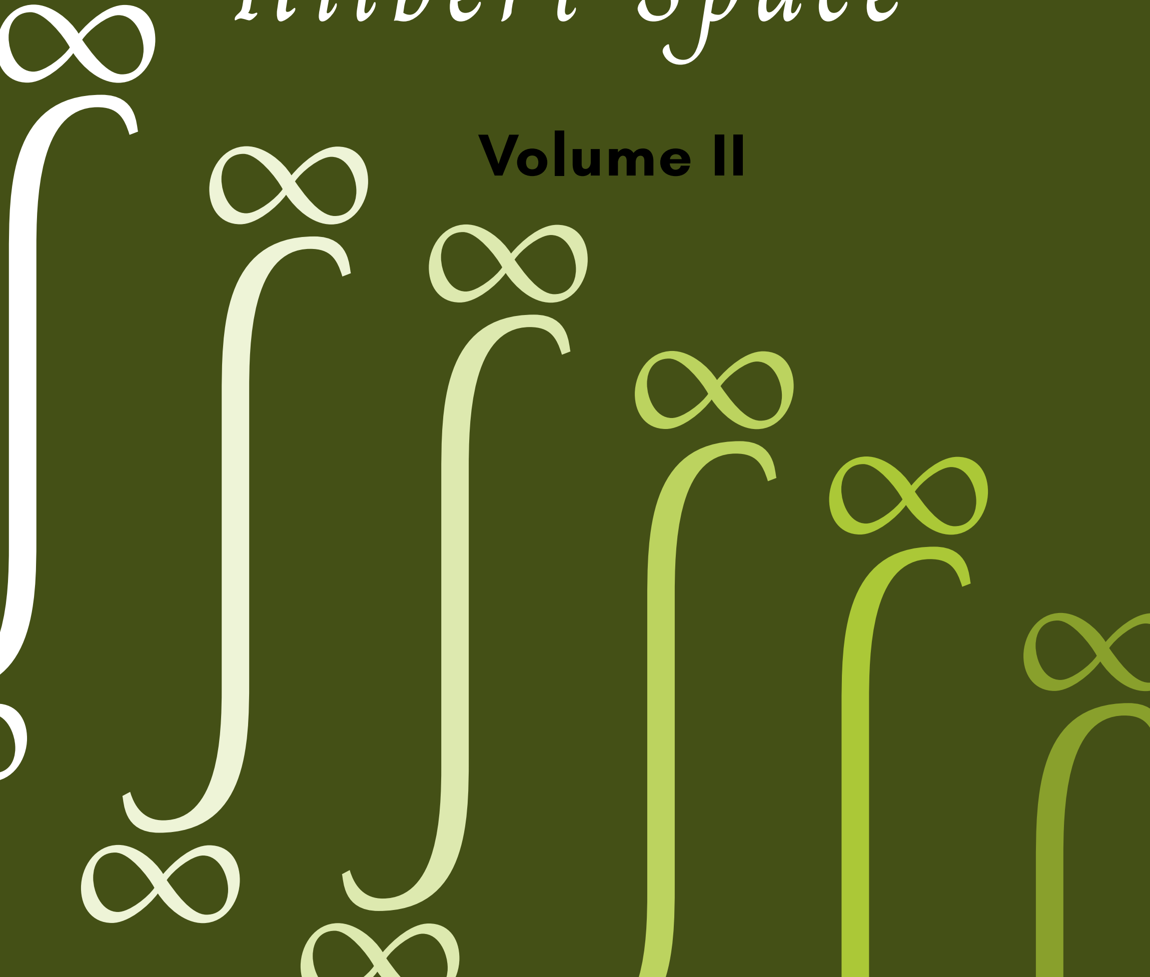
N.I. Akhiezer, I.M. Glazman

Theory of Linear Operators in Hilbert Space

Volume II



THEORY OF LINEAR OPERATORS IN HILBERT SPACE

N. I. Akhiezer and I. M. Glazman

THEORY OF LINEAR OPERATORS IN HILBERT SPACE

(Teoriia lineinykh operatorov v Gil'bertovom prostranstve,
Chapter VI to Appendix II)

Translated from the Russian by
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Chapter VI

THE SPECTRAL ANALYSIS OF UNITARY AND SELF-ADJOINT OPERATORS

58. The Trigonometric Moment Problem

Our next considerations will be based on a series of facts related to the so-called *moment problem*. In general form, the moment problem can be formulated as follows: let there be given a set of functions $u_\alpha(t)$ $(a \leqq t \leqq b)$ and a set of numbers c_α; the parameter α determines a one-to-one correspondence between these sets; it is required to find a nondecreasing function of bounded variation $\sigma(t)$ $(a \leqq t \leqq b)$ satisfying the system of equations

$$\int_a^b u_\alpha(t)\, d\sigma(t) = c_\alpha. \tag{1}$$

This problem consists of several parts, of which the basic one is the determination of conditions for the solvability of the above system of equations in the indicated class of functions $\sigma(t)$.

The earliest moment problem, which concerned the important special case with the given set of functions

$$1, t, t^2, \ldots,$$

(*the algebraic moment problem*) was studied by P. L. Tchebysheff and A. A. Markov in their remarkable works on limiting values of integrals.

In the present section we consider the trigonometric moment problem. In this case the system (1) has the form

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \quad (\pm k = 0, 1, 2, \ldots;\ \bar{c}_k = c_{-k}). \tag{2}$$

THEOREM: *For the existence of a nondecreasing function*[1] $\sigma(t)$ *satisfying equations* (2) *it is* (a) *necessary that the non-negativity of the trigonometric sum*

$$\sum_{k=-n}^{n} \xi_k e^{ikt} \qquad (n = 0, 1, 2, \ldots)$$

in the whole interval $[0, 2\pi]$ *implies the inequality*

---

[1] The boundedness of the variation follows automatically from equation (2) for $k = 0$.

$$\sum_{k=-n}^{n} \xi_k c_k \geqq 0$$

and, (b) *sufficient that for any real number u the expressions*

$$\sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) e^{-iku} c_k \qquad (n = 1, 2, 3, \ldots)$$

be non-negative, which correspond to the particular trigonometric sums

$$\sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) e^{ik(t-u)} = \frac{1}{n} \left| \sum_{k=0}^{n-1} e^{ik(t-u)} \right|^2 .$$

Proof: If the system (2) has a solution $\sigma(t)$, then

$$\sum_{k=-n}^{n} \xi_k c_k = \int_0^{2\pi} \sum_{k=-n}^{n} \xi_k e^{ikt} d\sigma(t),$$

and this yields the necessity of the first condition stated in the theorem. In order to prove the second part of the theorem, we consider a sequence of trigonometric sums

$$\psi_n(u) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) c_k e^{-iku} \qquad (n = 1, 2, 3, \ldots),$$

which are non-negative by hypothesis. Let

$$\sigma_n(t) = \frac{1}{2\pi} \int_0^t \psi_n(u)\, du \qquad (n = 1, 2, 3, \ldots).$$

These are nondecreasing functions for which

$$\sigma_n(0) = 0,\ \sigma_n(2\pi) = c_0 \qquad (n = 1, 2, 3, \ldots).$$

Thus, we have a sequence of functions $\sigma_n(t)$ which are nondecreasing in the interval $[0, 2\pi]$ and satisfy the inequality

$$0 \leqq \sigma_n(t) \leqq c_0 \qquad (n = 1, 2, 3, \ldots).$$

By the first theorem of Helly there exists a nondecreasing function $\sigma(t)$ and a subsequence $\{\sigma_{n_j}(t)\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \sigma_{n_j}(t) = \sigma(t)$$

at each point of continuity of $\sigma(t)$. By the second theorem of Helly

$$\int_0^{2\pi} e^{ikt} d\sigma(t) = \lim_{j\to\infty} \int_0^{2\pi} e^{ikt} d\sigma_{n_j}(t) = \lim_{j\to\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{ikt} \psi_{n_j}(t)\, dt =$$

$$= \lim_{j\to\infty} \left(1 - \frac{|k|}{n_j}\right) c_k = c_k,$$

and our assertion is completely proved.

We turn now to the question of the number of solutions of the system (2) under the assumption that it is solvable. Let us assume that (2) has two distinct nondecreasing solutions, $\sigma(t)$ and $\sigma^*(t)$. Their difference

$$\omega(t) = \sigma(t) - \sigma^*(t)$$

is a function of bounded variation for which

$$\int_0^{2\pi} e^{ikt} d\,\omega(t) = 0 \qquad (\pm k = 0, 1, 2, \ldots). \tag{3}$$

Now we shall prove that if the (real or even complex) function of bounded variation $\omega(t)$ satisfies (3), then it is constant at all its points of continuity. With this aim, we consider, for $k = \pm 1, \pm 2, \ldots$, the equation

$$\int_0^{2\pi} e^{ikt} d\,\omega(t) = e^{ikt}\omega(t)\Big|_0^{2\pi} - ik\int_0^{2\pi} e^{ikt}\,\omega(t)\,dt =$$

$$= \int_0^{2\pi} d\,\omega(t) - ik\int_0^{2\pi} e^{ikt}\omega(t)\,dt.$$

It follows by means of (3) that

$$\int_0^{2\pi} e^{ikt}\omega(t)\,dt = 0 \qquad (\pm k = 1, 2, 3, \ldots).$$

Letting

$$C = \frac{1}{2\pi}\int_0^{2\pi} \omega(t)\,dt,$$

we find that

$$\int_0^{2\pi} e^{ikt}\{\omega(t) - C\}\,dt = 0 \qquad (\pm k = 0, 1, 2, \ldots).$$

It follows from this equation and a fundamental uniqueness theorem from the theory of Fourier series that

$$\omega(t) - C = 0$$

at each point of continuity of $\omega(t)$.

From what has been proved, it follows, in particular, that, if $\sigma(t)$ and $\sigma^*(t)$ are two distinct solutions of (2), then their difference is constant at all the points where this difference is continuous. We say, therefore, that the solution of (2) is *essentially* unique. If we impose certain normalization conditions on the function $\sigma(t)$, then the solution is unique without reservation, i.e., the italicized word "essentially" is unnecessary. These normalization conditions involve, first, the points of discontinuity of $\sigma(t)$ and, second, the end points of the interval $[0, 2\pi]$. At each point t of discontinuity in the open interval $(0, 2\pi)$, the limits $\sigma(t-0)$ and $\sigma(t+0)$ exist. The value at the point t itself can be any number lying between these limits. We let

$$\sigma(t) = \sigma(t-0),$$

i.e., we require that the function $\sigma(t)$ be continuous from the left. In order to motivate further normalization conditions, we point out that the function $\sigma(t)$ itself is of no interest; rather, the interest is in the integrals

$$\int_0^{2\pi} f(t)\, d\sigma(t),$$

where $f(t)$ is a continuous function with period 2π. Every integral of this kind may be represented in the form

$$\int_0^{2\pi} f(t)\, d\sigma(t) = \int_{+0}^{2\pi-0} f(t)\, d\sigma(t) + \\ + f(2\pi)\{\sigma(2\pi) - \sigma(2\pi-0)\} + f(0)\{\sigma(+0) - \sigma(0)\}.$$

Since

$$f(2\pi) = f(0)$$

we have

$$\int_0^{2\pi} f(t) d\sigma(t) = \int_{+0}^{2\pi-0} f(t) d\sigma(t) + f(0)\{\sigma(2\pi) - \sigma(2\pi-0) + \sigma(+0) - \sigma(0)\} = \\ = \int_0^{2\pi-0} f(t)\, d\sigma^*(t) = \int_0^{2\pi} f(t)\, d\sigma^*(t),$$

where

$$\sigma^*(t) = \sigma(t) \qquad (0 < t < 2\pi),$$
$$\sigma^*(2\pi) = \sigma^*(2\pi-0) = \sigma(2\pi-0),$$
$$\sigma^*(+0) - \sigma^*(0) = \sigma(+0) - \sigma(0) + \sigma(2\pi) - \sigma(2\pi-0).$$

Thus, the new function $\sigma^*(t)$ is continuous from the left also at the point 2π. At the point zero it has a well-determined jump from the right (*concentrated mass*) which in special cases can equal zero. Finally, let us define

$$\tilde{\sigma}(t) = \sigma^*(t) - \sigma^*(0).$$

Then the equation

$$\int_0^{2\pi} f(t)\, d\sigma(t) = \int_0^{2\pi} f(t)\, d\tilde{\sigma}(t)$$

holds for any continuous periodic function $f(t)$. Both $\tilde{\sigma}(t)$ and $\sigma^*(t)$ are continuous from the left in the interval $0 < t \leqq 2\pi$. But, in addition, $\tilde{\sigma}(0) = 0$. We will always impose these normalization conditions on the solutions of the trigonometric moment problem. They serve to determine a unique solution.

59. Analytic Functions with Values in a Half-plane

In the present section we consider analytic functions, regular in the interior of a circle or a half-plane, which take on values which lie in some half-plane.

THEOREM 1: *In order that a given function $f(\zeta)$, finite in the circle $|\zeta| < 1$, admit the representation*

$$f(\zeta) = i\beta + \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta}\, d\sigma(t), \tag{1}$$

where β is a real constant and $\sigma(t)$ is a nondecreasing function[2], *it is necessary and sufficient that $f(\zeta)$ be regular and have a non-negative real part in the circle $|\zeta| < 1$.*

PRELIMINARY REMARK. We have here a particular moment problem with the moment function

$$u_\zeta(t) = \frac{e^{it} + \zeta}{e^{it} - \zeta}.$$

The role of the moments is played by the numbers

$$c_\zeta = f(\zeta) - i\beta,$$

where the parameter ζ runs through a discrete sequence in the circle $|\zeta| < 1$.

We remark also that $\beta = \Im f(0)$.

Proof: The necessity of the condition of the theorem is obvious, since the right member of formula (1) is regular in the region $|\zeta| < 1$ and

$$\Re\left\{\int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta}\, d\sigma(t)\right\} = \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2}\, d\sigma(t) \geqq 0,$$

where $r = |\zeta| < 1$ and $\varphi = \arg \zeta$.

---

[2] The boundedness of the variation of $\sigma(t)$ is obtained automatically from the finiteness of the value $f(0)$.

We prove now that the condition is sufficient. Assume that $f(\zeta)$ is regular and has non-negative real part in the circle $|\zeta| < 1$. Then, as is known from the theory of functions, $f(\zeta)$ is given, for $|\zeta| \leqq R < 1$, by

$$f(\zeta) = i\beta + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{it} + \zeta}{Re^{it} - \zeta} u(Re^{it})\, dt,$$

where

$$u(re^{it}) = \frac{f(re^{it}) + \overline{f(re^{it})}}{2}$$

is the real part of the function $f(\zeta)$. Hence

$$\Re f(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{it})\, dt.$$

The indicated representation can be written in the form

$$f(\zeta) = i\beta + \int_0^{2\pi} \frac{Re^{it} + \zeta}{Re^{it} - \zeta}\, d\sigma_R(t), \tag{2}$$

where

$$\sigma_R(t) = \frac{1}{2\pi}\int_0^{t} u(Re^{is})\, ds.$$

Since $u(Re^{is}) \geqq 0$ by assumption, $\sigma_R(t)$ is a nondecreasing function of t and, for $0 \leqq t \leqq 2$,

$$0 \leqq \sigma_R(t) \leqq \sigma_R(2\pi) = \Re f(0).$$

Thus, the set of functions $\sigma_R(t)$ $(0<R<1)$ is uniformly bounded. By the Helly theorem, there exists a nondecreasing function $\sigma(t)$ and a sequence

$$R_1 < R_2 < R_3 < \dots \qquad (R_j \to 1),$$

such that

$$\lim_{j\to\infty} \sigma_{R_j}(t) = \sigma(t)$$

at each point of continuity of $\sigma(t)$. Applying the second Helly theorem to (2), we find that

$$f(\zeta) = i\beta + \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta}\, d\sigma(t) \tag{3}$$

for each ζ in the circle $|\zeta| < 1$. Thus, the theorem is proved.

If we expand $f(\zeta)$ in a Maclaurin series

$$f(\zeta) = c + 2c_{-1}\zeta + 2c_{-2}\zeta^2 + \dots$$

and let $\frac{c+\bar{c}}{2} = c_0$, then by (3) we obtain the following expressions for the coefficients

$$c_0 = \int_0^{2\pi} d\,\sigma(t)\,,$$

$$c_{-k} = \int_0^{2\pi} e^{-ikt} d\,\sigma(t) \qquad (k = 1, 2, 3, \ldots).$$

Introducing the numbers $c_k = \bar{c}_{-k} (k = 1, 2, 3, \ldots)$, we obtain the equations which appear in the trigonometric moment problem. Therefore, under the normalization conditions of the preceding section, the function $\sigma(t)$ in (3) is uniquely determined by $f(\zeta)$.

This fact can be proved also with the aid of the following well-known inversion formula from the theory of Fourier series.

THEOREM 2: (*R. Nevanlinna*) *In order that a given function* $\varphi(z)$, *finite in the half-plane* $\Im z > 0$, *admit the representation*

$$\varphi(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\,\sigma(t)$$

(*where* $\mu \geqq 0$ *and* α *are two real constants, and* $\sigma(t)$ *is a nondecreasing function*[3]), *it is necessary and sufficient that* $\varphi(z)$ *be regular and have non-negative imaginary part in the half-plane* $\Im z > 0$. *Here the Stieltjes integral with infinite limits of integration is to be interpreted as*

$$\lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B \frac{1+tz}{t-z} d\,\sigma(t),$$

which corresponds to the assumption that the limits

$$\sigma(-\infty) = \lim_{A \to -\infty} \sigma(A), \quad \sigma(\infty) = \lim_{B \to \infty} \sigma(B)$$

are finite. If, in addition, one applies the normalization conditions

$$\sigma(t-0) = \sigma(t), \quad \sigma(-\infty) = 0,$$

then the function $\sigma(t)$ *is uniquely determined.*

Proof: Let

$$z = i\frac{1+\zeta}{1-\zeta},$$

$$\varphi(z) = if(\zeta).$$

---

[3] The boundedness of the variation of $\sigma(t)$ follows from the finiteness of the value $\varphi(i)$.

These transformations map the half-plane $\Im z > 0$ into the circular region $|\zeta| < 1$, and the function $\varphi(z)$, regular in the half-plane $\Im z > 0$, into a function $f(\zeta)$, regular and having non-negative real part in the circle. The integral representation of such a function was treated above. It has the form

$$f(\zeta) = i\beta + \int_0^{2\pi} \frac{e^{is} + \zeta}{e^{is} - \zeta} d\rho(s) = i\beta + \int_{+0}^{2\pi-0} \frac{e^{is} + \zeta}{e^{is} - \zeta} d\rho(s) +$$

$$+ \frac{1+\zeta}{1-\zeta}\{\rho(2\pi) - \rho(2\pi - 0) + \rho(+0) - \rho(0)\} =$$

$$= i\beta + \mu\frac{1+\zeta}{1-\zeta} + \int_{+0}^{2\pi-0} \frac{e^{is} + \zeta}{e^{is} - \zeta} d\rho(s),$$

where we denote the distribution by $\rho(s)$ instead of $\sigma(s)$. With the aid of the transformation defined above we obtain

$$\varphi(z) = -\beta + i\mu\frac{1+\zeta}{1-\zeta} + i\int_{+0}^{2\pi-0} \frac{e^{is} + \zeta}{e^{is} - \zeta} d\rho(s) \tag{4}$$

or

$$\varphi(z) = -\beta + \mu z + \int_{+0}^{2\pi-0} \frac{z\cot\frac{s}{2} - 1}{\cot\frac{s}{2} + z} d\rho(s).$$

Letting

$$-\beta = \alpha, \quad -\cot\frac{s}{2} = t, \quad \rho(s) = \sigma(t),$$

we obtain for $\varphi(z)$ the representation

$$\varphi(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma(t). \tag{5}$$

The transition from (4) to (5) is also correct in the reverse direction. Therefore, Theorem 2 is proved.

THEOREM 3: *In order that a function $\varphi(z)$ in the half-plane $\Im z > 0$ admit the representation*

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{t-z}$$

with a nondecreasing function of bounded variation $\omega(t)$, it is necessary *and sufficient that* $\varphi(z)$ *be regular and have non-negative imaginary part in the half-plane* $\Im z > 0$ *and*

$$\sup_{y \to \infty} | y \varphi(iy) | < \infty.$$

The function $\omega(t)$ *is uniquely determined by* $\varphi(z)$ *if we require that*

$$\omega(-\infty) = \lim_{A \to -\infty} \omega(A) = 0, \quad \omega(t-0) = \omega(t) \quad (-\infty < t \leqq \infty).$$

Proof: The necessity of the condition is easily verified. We concern ourselves only with the proof of the sufficiency. If the function $\varphi(z)$ is regular and has non-negative imaginary part in the half-plane $\Im z > 0$, then it admits in every case the representation

$$\varphi(z) = \alpha + \mu z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\sigma(t),$$

as we saw in the previous theorem. From this representation it follows that

$$y\varphi(iy) = \alpha y + i\mu y^2 + \int_{-\infty}^{\infty} \frac{y(1 + ity)}{t - iy} \, d\sigma(t).$$

By hypothesis, there exists a constant M such that

$$\left| \alpha y + i\mu y^2 + \int_{-\infty}^{\infty} \frac{y(1 + ity)}{t - iy} \, d\sigma(t) \right| \leqq \mathrm{M} \qquad (y > 0).$$

Hence, a fortiori,

$$(6) \qquad \begin{cases} \left| \alpha y + \displaystyle\int_{-\infty}^{\infty} \frac{y(1 - y^2)t}{t^2 + y^2} \, d\sigma(t) \right| \leqq \mathrm{M}, \\ \left| \mu y^2 + y^2 \displaystyle\int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + y^2} \, d\sigma(t) \right| \leqq \mathrm{M}. \end{cases}$$

The second inequality shows that $\mu = 0$ and also that

$$\int_{-\infty}^{\infty} \frac{y^2}{t^2 + y^2} (1 + t^2) \, d\sigma(t) \leqq \mathrm{M}.$$

Hence, for any $N > 0$,

$$\int_{-N}^{N} \frac{y^2}{t^2 + y^2} (1 + t^2) \, d\sigma(t) \leqq \mathrm{M}.$$

Let $y \to \infty$ to obtain

$$\int_{-N}^{N} (1 + t^2)\, d\sigma(t) \leqq \mathrm{M},$$

and, consequently,

$$\int_{-\infty}^{\infty} (1 + t^2)\, d\sigma(t) \leqq \mathrm{M}.$$

We can, therefore, introduce the nondecreasing function

$$\omega(t) = \int_{-\infty}^{t} (1 + t^2)\, d\sigma(t),$$

for which it is obvious that

$$\lim_{t \to -\infty} \omega(t) = 0,$$

and which, together with $\sigma(t)$, is continuous from the left.

From the first inequality in (6) it follows that

$$a = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{(y^2 - 1)\, t}{t^2 + y^2}\, d\sigma(t),$$

From this one easily gets

$$a = \int_{-\infty}^{\infty} t\, d\sigma(t).$$

Since, as was shown, $\mu = 0$, (5) takes the form

$$\varphi(z) = \int_{-\infty}^{\infty} t d\sigma(t) + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t)$$

or

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{t - z},$$

which was to be established.

The uniqueness of this representation (under our normalization conditions) is a consequence of the uniqueness of the function $\sigma(t)$ in Theorem 2, since this follows from the equation

$$\sigma(t) = \int_{-\infty}^{t} \frac{d\omega(t)}{1 + t^2},$$

which relates the functions $\sigma(t)$ and $\omega(t)$. However, it is possible to establish the uniqueness directly by using the following Stieltjes inversion formula:

$$\frac{\omega(t-0)+\omega(t+0)}{2} = \text{const} + \lim_{y\to 0}\frac{1}{\pi}\int_0^t \Im\,\varphi(x+iy)\,dx.$$

60. The Theorem of Bochner

In the present section we consider the continuous analogue of the trigonometric moment problem. The problem consists of finding conditions, which a given function $F(t)$, $-\infty < t < \infty$, must satisfy in order that it have a representation

$$F(t) = \int_{-\infty}^{\infty} e^{ist}\,d\omega(s), \tag{1}$$

where $\omega(s)$ is a nondecreasing function.[4]

Bochner was the first to prove that such a representation exists and is unique under suitable normalization[5] if and only if the function $F(t)$ is continuous and

$$\sum_{\alpha,\beta=1}^{n} F(t_\alpha - t_\beta)\rho_\alpha\bar{\rho}_\beta \geqq 0, \tag{2}$$

for any integer n, any real $t_1, t_2, \ldots, t_n$, and any complex $\rho_1, \rho_2, \ldots, \rho_n$. Functions which satisfy (2) are called positive definite.

The necessity of these conditions is easily verified. In fact, the right side of (1) is a continuous function of t, and, on the other hand, by (1),

$$\sum_{\alpha,\beta=1}^{n} F(t_\alpha - t_\beta)\rho_\alpha\bar{\rho}_\beta = \int_{-\infty}^{\infty} \Big|\sum_{k=1}^{n}\rho_k e^{ist_k}\Big|^2 d\omega(s),$$

the right side of which is finite and non-negative.

The proof of the sufficiency, to which we now turn, is not so simple. Let $F(t)$ be a continuous function which satisfies (2). We remark that, by (2), for any t $(-\infty < t < \infty)$ the equation

$$\overline{F(t)} = F(-t)$$

and the inequality

$$|F(t)| \leqq F(0) \tag{3}$$

both hold. Let

---

[4] The boundedness of its variation follows from the finiteness of the value $F(0)$.

[5] cf. Theorem 3 of the preceding section.

$$\Phi(z) = \int_0^\infty e^{itz} F(t)\,dt.$$

By (3), the function $\Phi(z)$ is regular in the region $\Im z > 0$. In addition, for each $y > 0$,

$$|y\,\Phi(iy)| \leqq \int_0^\infty ye^{-ty}\,|F(t)|\,dt \leqq F(0)\int_0^\infty ye^{-ty}\,dt = F(0).$$

Now we show that

$$\Re\,\Phi(z) \geqq 0 \qquad (z = x + iy)$$

for $y > 0$. With this aim let us introduce the identity

$$\frac{1}{2y} = \int_0^\infty e^{izv}\,e^{-i\bar{z}v}\,dv.$$

We have further

$$\Phi(z) + \overline{\Phi(z)} = \int_0^\infty e^{izu}\,F(u)\,du + \int_0^\infty e^{-i\bar{z}u}\,F(-u)\,du.$$

Therefore,

$$\frac{\Phi(z) + \overline{\Phi(z)}}{2y} = \int_0^\infty\int_0^\infty e^{iz(u+v)}\,e^{-i\bar{z}v}\,F(u)\,du\,dv + \int_0^\infty\int_0^\infty e^{-i\bar{z}(u+v)}\,e^{izv}F(-u)\,du\,dv =$$

$$= \int_0^\infty d\beta\int_\beta^\infty e^{i\alpha z}\,e^{-i\beta\bar{z}}F(\alpha - \beta)\,d\alpha + \int_0^\infty d\alpha\int_\alpha^\infty e^{i\alpha z}\,e^{-i\beta\bar{z}}\,F(\alpha-\beta)\,d\beta =$$

$$= \int_0^\infty\int_0^\infty F(\alpha - \beta)\,e^{ix(\alpha-\beta)}\,e^{-y(\alpha+\beta)}\,d\alpha\,d\beta =$$

$$= \lim_{A\to\infty}\int_0^A\int_0^A F(\alpha - \beta)\,e^{ix(\alpha-\beta)}\,e^{-y(\alpha+\beta)}\,d\alpha\,d\beta =$$

$$= \lim_{A\to\infty}\lim_{n\to\infty}\frac{A^2}{n^2}\sum_{r,s=1}^{n} F\left(\frac{r-s}{n}A\right)e^{ix(r-s)A/n}\,e^{-y(r+s)A/n},$$

and our assertion follows from the fact that the sum on the right side has the form (2) (it is sufficient to let $t_k = kA/n$, $\rho_k = e^{-y(kA/n)}\,e^{ix(kA/n)}$).

The function $\Phi(z)$ differs only by the factor i from a function considered in the previous section (cf. Theorem 3). Hence, under our normalization conditions, there exists a unique nondecreasing function $\omega(s)$ such that

$$\Phi(z) = i\int_{-\infty}^{\infty} \frac{d\omega(s)}{s+z} \qquad (\Im z > 0).$$

On the other hand,

$$\frac{i}{s+z} = \int_{0}^{\infty} e^{i(s+z)t}\, dt$$

and, hence,

$$\int_{0}^{\infty} e^{itz} F(t)\, dt = \int_{-\infty}^{\infty} d\omega(s) \int_{0}^{\infty} e^{i(s+z)t}\, dt = \int_{0}^{\infty} e^{itz}\, dt \int_{-\infty}^{\infty} e^{ist}\, d\omega(s).$$

This means that two piece-wise continuous absolutely integrable functions of t, which equal zero for $t < 0$ and equal

$$e^{-ty} F(t), \ e^{-ty} \int_{-\infty}^{\infty} e^{ist}\, d\omega(s)$$

for $t \geqq 0$, have identical Fourier transforms. By a well-known uniqueness theorem these functions are identical. Hence, the representation (1) is obtained and its uniqueness is proved.

From the uniqueness of this representation we have the following more general proposition: if $\omega_1(s)$ and $\omega_2(s)$ are two complex functions of bounded variation which are normalized in the usual manner and if, for $-\infty < t < \infty$,

$$\int_{-\infty}^{\infty} e^{ist}\, d\omega_1(s) = \int_{-\infty}^{\infty} e^{ist}\, d\omega_2(s), \tag{4}$$

then

$$\omega_1(s) = \omega_2(s).$$

Indeed, if (4) holds and

$$\tilde{\omega}(s) = \omega_1(s) - \omega_2(s),$$

then

$$\int_{-\infty}^{\infty} e^{ist}\, d\tilde{\omega}(s) = 0 \qquad (-\infty < t < \infty).$$

If we let

$$\tilde{\omega}(s) = \varphi(s) + i\psi(s),$$

where $\varphi(s)$ and $\psi(s)$ are real, then we get

$$\int_{-\infty}^{\infty} \cos st \, d\varphi(s) = \int_{-\infty}^{\infty} \sin st \, d\psi(s),$$

$$\int_{-\infty}^{\infty} \sin st \, d\varphi(s) = -\int_{-\infty}^{\infty} \cos st \, d\psi(s).$$

Since one member of each of the equations is an even function, and the other members are odd functions,

$$\int_{-\infty}^{\infty} \cos st \, d\varphi(s) = \int_{-\infty}^{\infty} \sin st \, d\varphi(s) = \int_{-\infty}^{\infty} \cos st \, d\psi(s) = \int_{-\infty}^{\infty} \sin st \, d\psi(s) = 0.$$

In other words, for all real t,

$$\int_{-\infty}^{\infty} e^{ist} \, d\varphi(s) = 0, \quad \int_{-\infty}^{\infty} e^{ist} \, d\psi(s) = 0.$$

Since each of the functions $\varphi(s)$ and $\psi(s)$ is the difference of two nondecreasing functions, the equations

$$\varphi(s) = 0, \; \psi(s) = 0 \qquad (-\infty < s < \infty)$$

are direct consequences of Bochner's theorem.

61. The Resolution of the Identity

We recall the results of Section 55 concerning completely continuous self-adjoint operators. Here we assume that the operator—we denote it by A—operates in a Hilbert space H (rather than in an arbitrary inner product space R). As we have seen, the operator A has a finite or countable set of pairwise orthogonal and normalized eigenvectors $e_1, e_2, e_3, \ldots$, which correspond to nonzero eigenvalues. The set H_0 of all vectors from H which are orthogonal to each vector e_k is a subspace. From the theorem of section 55 it follows that for $h \in H_0$, $Ah = 0$. Therefore, it is possible to consider H_0 as the eigenmanifold of the operator A which belongs to the eigenvalue $\lambda = 0$. This space is separable if H is separable, and not separable otherwise. As far as the subspace $H \ominus H_0$ is concerned, it is always separable and $\{e_k\}$ is an orthonormal basis for it. We can represent the subspace $H \ominus H_0$ in the form of an orthogonal sum of eigenmanifolds H_k, which belong to distinct eigenvalues. Then

$$H = H_0 \oplus H_1 \oplus H_2 \oplus \ldots,$$

where

$$Af = \lambda_k f$$

for $f \in H_k$ $(\lambda_0 = 0)$.

Thus, we have decomposed the space H into subspaces, in each of which the operation of A is merely the multiplication of an element by a particular number. Designating the operator of projection on H_k by P_k we can write

$$E = P_0 + P_1 + P_2 + \dots . \tag{1}$$

For each $f \in H$,

$$Af = \lambda_1 P_1 f + \lambda_2 P_2 f + \dots . \tag{2}$$

The numbers λ_k appearing here are real, they lie in a finite interval, and can have as a limit point only the single point 0. To enumerate the numbers λ_k in the order of increasing or decreasing magnitude is possible only if they are finite in number or if all are of one sign. Therefore, the terms of the series (2) follow not in their natural order, but in the order determined by the enumeration of the set of eigenvalues. This defect of the representations (1) and (2) can be removed by using the Stieltjes integral.

With this aim let us introduce, for each real t, the subspace G_t spanned by the eigenvectors which belong to eigenvalues less than t. Here we regard zero as the eigenvalue which belongs to the eigenmanifold H_0. Let E_t be the operator of projection on G_t. The operator E_t has a limit both for increasing and for decreasing t. Therefore, E_{t-0} and E_{t+0} exist. It is easy to see that

$$E_{t-0} = E_t.$$

Thus, E_t is a function of t which is continuous from the left. If λ_k is an eigenvalue, then the difference

$$E_{\lambda_k+0} - E_{\lambda_k} = P_k$$

is the operator of projection on the eigenmanifold H_k. Now formulas (1) and (2) can be represented in the form

$$f = Ef = \int_{\alpha}^{\beta} d E_t f, \tag{1'}$$

$$Af = \int_{\alpha}^{\beta} t \, d E_t f, \tag{2'}$$

where the integrals are taken over an interval $[\alpha, \beta]$ which contains all the eigenvalues of the operator A.

These integrals are nothing other than sums of particular series. We have introduced the integral representations because, as we shall see later, they generalize to arbitrary (not necessarily completely continuous) self-adjoint operators in H. Anticipating this generalization we make the following definition.

DEFINITION: *A resolution of the identity is a one-parameter family of projection operators* E_t, *where* t *runs through a finite or infinite interval* $[\alpha, \beta]$, *which satisfy the following conditions.*[6]

$$\text{(a)} \quad E_\alpha = 0, \ E_\beta = E,$$

$$\text{(b)} \quad E_{t-0} = E_t \quad (\alpha < t < \beta),$$

$$\text{(c)} \quad E_u E_v = E_s \quad (s = \min\{u, v\}).$$

From the definition it follows that, for each fixed $f \in \mathrm{H}$, the function of t given by

$$(E_t f, f) = \sigma(t)$$

is nondecreasing, has bounded variation, is continuous from the left, and

$$\sigma(\alpha) = 0, \ \sigma(\beta) = (f, f).$$

Indeed, for $s < t$,

$$(E_s f, f) = \| E_s f \|^2 = \| E_s E_t f \|^2 \leqslant \| E_t f \|^2 = (E_t f, f).$$

For each interval $\Delta = [t', t''] \subset [\alpha, \beta]$, we denote the difference $E_{t''} - E_{t'}$ by $E(\Delta)$. If Δ_1 and Δ_2 are intervals of this type, it follows from (c) that

$$E(\Delta_1) E(\Delta_2) = E(\Delta),$$

where Δ is the intersection of Δ_1 and Δ_2. In particular, if the intervals Δ_1 and Δ_2 do not have common interior points, then

$$E(\Delta_1) E(\Delta_2) = 0,$$

i.e., the subspaces on which $E(\Delta_1)$ and $E(\Delta_2)$ project are orthogonal. On the basis of what has been said, property (c) is called the *orthogonality property* of the decomposition of the identity.

The considerations which preceded our general definition shows that every completely continuous self-adjoint operator generates a resolution of the identity. In the next section we show that arbitrary unitary operators and arbitrary self-adjoint operators in H have well-defined resolutions of the identity. It is true that these resolutions of the identity do not have those simple properties which the resolution of the identity of completely continuous operators possess. This, as we will see, is connected with the fact that the spectra of these operators have more complicated structures than the spectra of completely continuous operators.

62. The Integral Representation of a Unitary Operator

Let U be a unitary operator in H. We choose an arbitrary element $f \in \mathrm{H}$ and let

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[6] If the interval $[\alpha, \beta]$ is infinite then we define $E_{-\infty} = \lim_{t \to -\infty} E_t$, $E_\infty = \lim_{t \to \infty} E_t$.

$$(U^kf, f) = c_k(f) = c_k \qquad (\pm k = 0, 1, 2, \ldots), \tag{1}$$

so that

$$c_{-k} = (U^{-k}f, f) = (f, U^kf) = \bar{c}_k.$$

We shall prove that

$$\Phi_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) c_k\, e^{-ikt} \geqq 0 \tag{2}$$

for each integer n and each real t. By (1),

$$\Phi_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) e^{-ikt}\, (U^kf, f).$$

On the other hand, if

$$T = E + e^{-it}U + e^{-2it}U^2 + \ldots + e^{-(n-1)it}U^{n-1},$$

then

$$(Tf, Tf) = \sum_{r,s=0}^{n-1} e^{i(s-r)t}(U^rf, U^sf) =$$

$$= \sum_{r,s=0}^{n-1} e^{i(s-r)t}(U^{r-s}f, f) = \sum_{k=-n}^{n} (n - |k|)\, e^{-ikt}(U^kf, f).$$

Therefore,

$$\Phi_n(t) = \frac{1}{n}(Tf, Tf),$$

which implies (2).

The sequence $\{c_k\}_{-\infty}^{\infty}$ satisfies the conditions of the theorem of Section 58. Therefore, there is a uniquely defined normalized nondecreasing function $\sigma(t)$ for which

$$c_k = \int_0^{2\pi} e^{ikt} d\sigma(t) \qquad (\pm k = 0, 1, 2, \ldots).$$

For each fixed t, the function $\sigma(t)$ is some functional of f:

$$\sigma(t) = \sigma(t; f).$$

With the aid of this functional, we now define another functional dependent on a pair of vectors $f, g \in \mathrm{H}$:

$$\sigma(t; f, g) = \frac{1}{4}\sigma(t; f + g) - \frac{1}{4}\sigma(t; f - g) +$$

$$+ \frac{i}{4}\sigma(t; f + ig) - \frac{i}{4}\sigma(t, f - ig).$$

As above, t is any real number in the interval $[0, 2\pi]$. In the equation

$$\int_0^{2\pi} e^{ikt} d\sigma(t; h) = (U^k h, h) \qquad (\pm k = 0, 1, 2, \ldots)$$

substitute, in turn,

$$h = f + g,\ f - g,\ f + ig,\ f - ig.$$

to obtain

$$\int_0^{2\pi} e^{ikt} d\sigma(t; f, g) = (U^k f, g) \qquad (\pm k = 0, 1, 2, \ldots). \tag{3}$$

Thus, we have derived representations in the form of trigonometric Stieltjes integrals not only for the quantity $(U^k f, f)$ but, more generally, for $(U^k f, g)$. This representation is unique (under the normalization conditions on the function $\sigma(t; f, g)$) according to the general theorem at the end of Section 58.

Starting from the uniqueness of the representation (3) we prove that $\sigma(t; f, g)$ is a bilinear functional of f, g, whose norm does not exceed 1. Let

$$f = \alpha_1 f_1 + \alpha_2 f_2.$$

Then

$$(U^k f, g) = \alpha_1 (U^k f_1, g) + \alpha_2 (U^k f_2, g),$$

and, hence,

$$\int_0^{2\pi} e^{ikt} d\sigma(t; f, g) = \alpha_1 \int_0^{2\pi} e^{ikt} d\sigma(t; f_1, g) + \alpha_2 \int_0^{2\pi} e^{ikt} d\sigma(t; f_2, g) =$$

$$= \int_0^{2\pi} e^{ikt} d_t \{\alpha_1 \sigma(t; f_1, g) + \alpha_2 \sigma(t; f_2, g)\}.$$

Since this relation holds for $\pm k = 0, 1, 2, \ldots$, and the normalization condition is satisfied, we have

$$\sigma(t; f, g) = \alpha_1 \sigma(t; f_1, g) + \alpha_2 \sigma(t; f_2, g).$$

We see that σ, as a function of f, g, is linear in the first of these arguments.

We remark now that on one hand

$$(g, U^k f) = (U^{-k} g, f) = \int_0^{2\pi} e^{-ikt} d\sigma(t; g, f),$$

and on the other

$$(g, U^k f) = \overline{(U^k f, g)} = \int_0^{2\pi} e^{-ikt} d\overline{\sigma(t; f, g)}.$$

Therefore, for any integer k,

$$\int_0^{2\pi} e^{-ikt} d\overline{\sigma(t; f, g)} = \int_0^{2\pi} e^{-ikt} d\sigma(t; g, f).$$

Hence,

$$\overline{\sigma(t; f, g)} = \sigma(t; g, f).$$

From this relation and the linearity in the argument f, it follows that

$$\sigma(t; f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \sigma(t; f, g_1) + \bar{\beta}_2 \sigma(t; f, g_2).$$

We recall now the theorem of Section 20. Since $\sigma(t; f, g)$ is a non-decreasing function of t and since

$$\sigma(0; f, f) = 0,$$

$$\sigma(t; f, f) \leqq \sigma(2\pi, f, f) = \int_0^{2\pi} d\sigma(t; f, f) = (f, f),$$

it follows from the indicated theorem that $\sigma(t; f, g)$ is a bilinear functional of f, g with norm not exceeding 1.

By the theorem on the general form of a bilinear functional, there exists a family of operators E_t, which depend on the parameter $t (0 \leqq t \leqq 2\pi)$, such that

$$\sigma(t; f, g) = (E_t f, g).$$

Now we prove that E_t is a resolution of the identity.

It follows by means of the equation

$$\overline{\sigma(t; f, g)} = \sigma(t; g, f),$$

that

$$(g, E_t f) = (E_t g, f)$$

In other words, E_t is a bounded self-adjoint operator.

We write (3) in the form

$$(3') \qquad (U^k f, g) = \int_0^{2\pi} e^{ikt} d(E_t f, g),$$

and let

$$g = U^{-r} h \qquad (\pm r = 0, 1, 2, \ldots).$$

We obtain

$$(U^k f, U^{-r} h) = \int_0^{2\pi} e^{ikt} d(E_t f, U^{-r} h)$$

and, hence,

$$(U^{k+r} f, h) = \int_0^{2\pi} e^{ikt} d(U^r E_t f, h).$$

But since

$$(U^r E_t f, h) = \int_0^{2\pi} e^{irs} d_s(E_s E_t f, h),$$

we have

$$(U^{k+r} f, h) = \int_0^{2\pi} e^{ikt} d_t \left\{ \int_0^{2\pi} e^{irs} d_s(E_s E_t f, h) \right\} . \tag{4}$$

On the other hand,

$$(U^{k+r} f, h) = \int_0^{2\pi} e^{i(k+r)t} d(E_t f, h) = \int_0^{2\pi} e^{ikt} d_t \left\{ \int_0^{t} e^{irs} d(E_s f, h) \right\}. \tag{5}$$

From the validity of the representations (4), (5) for any integer k and from the normalization properties of the functions of t

$$\int_0^{2\pi} e^{irs} d_s(E_s E_t f, h), \quad \int_0^{t} e^{irs} d\,(E_s f, h),$$

it follows that

$$\int_0^{2\pi} e^{irs} d_s(E_s E_t f, h) = \int_0^{t} e^{irs} d\,(E_s f, h).$$

This relation is satisfied for any integer r. Therefore, by the uniqueness of the representation, the equation

$$(E_s E_t f, h) = (E_s f, h)$$

is correct for any $f, h \in \mathrm{H}$, whenever $s \leqq t$. We have also shown that

$$E_s E_t = E_s \qquad (s \leqq t), \tag{6}$$

which implies that

$$E_t^2 = E_t ,$$

i.e., E_t is a projection operator. Instead of (6), it is possible to write the more general relation

$$E_u E_v = E_s \qquad (s = \min \{u, v\}).$$

It remains for us to verify that

$$E_0 = O, \ E_{2\pi} = E, \ E_{t-0} = E_t \qquad (0 < t \leqq 2\pi)$$

in order to complete the proof that E_t is a resolution of the identity. The first and second equations need no proof. In order to prove the third relation, we use the normalization conditions which imply that

$$\lim_{\substack{s \to t \\ s < t}} (E_s f, g) = (E_t f, g).$$

Thus,

$$E_{t-0} = E_t$$

in the sense of weak convergence. But this equation also holds in the sense of strong convergence, since, for $t > s$,

$$\|(E_t - E_s)f\|^2 = ((E_t - E_s)f, f) = (E_t f, f) - (E_s f, f),$$

and this implies that

$$\|(E_t - E_s)f\|$$

tends to zero as $s \to t\ (s < t)$.

We have obtained a resolution of the identity, with the parameter interval $[0, 2\pi]$, which belongs to the operator U in the sense that for any integer k and any $f, g \in \mathrm{H}$

$$(3') \qquad (U^k f, g) = \int_0^{2\pi} e^{ikt} d(E_t f, g).$$

This equation is expressed often in the form

$$U^k f = \int_0^{2\pi} e^{ikt} dE_t f.$$

We show now *that this equation has a valid interpretation, in its own right, in addition to being a symbolic representation of* (3′). As in elementary integral calculus, we introduce a subdivision of the interval $[0, 2\pi]$:

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = 2\pi.$$

Corresponding to this subdivision we construct an operator[7]

$$T_n = e^{ikt_1}E(\Delta_1) + e^{ikt_2}E(\Delta_2) + \ldots + e^{ikt_n}E(\Delta_n),$$

where $\Delta_j = [t_{j-1}, t_j]$. Now we prove that, for each $f \in \mathrm{H}$,

$$(7) \qquad T_n f \to U^k f \qquad (n \to \infty),$$

if the largest of the lengths of the intervals Δ_j tends to zero. In other words, we prove that the sequence of operators T_n converges strongly to the operator U^k. After that, we use the limit of the operators T_n as the definition of the integral

$$\int_0^{2\pi} e^{ikt} dE_t,$$

In order to prove (7) we recall Section 23. In view of Theorem 1 of that section, we should verify that

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[7] Translator's Note: The dependence of T_n on k might have been indicated by writing T_{kn} in place of T_n.

(a) for any $f, g \in \mathrm{H}$

$$(T_n f, g) \to (U^k f, g) \qquad (n \to \infty),$$

(b) $$\| T_n f \| \to \| U^k f \| = \| f \|.$$

The validity of (a) is obvious from (3′), since

$$(T_n f, g) = \sum_{j=1}^{n} e^{ikt_j}(E(\Delta_j) f, g) \to \int_0^{2\pi} e^{ikt} d(E_t f, g).$$

In order to prove (b) we notice that

$$\| T_n f \|^2 = (T_n f, T_n f) = \sum_{r,s=1}^{n} e^{ik(t_r - t_s)}(E(\Delta_r) f, E(\Delta_s) f) =$$

$$= \sum_{r,s=1}^{n} e^{ik(t_r - t_s)}(E(\Delta_s) E(\Delta_r) f, f) = \sum_{r=1}^{n} (E(\Delta_r) f, f) = (f, f).$$

These relations show that not only does the limit equation (b) hold, but, simply,

$$\| T_n f \| = \| f \|.$$

From our considerations, we remark that the resolution of the identity $E_t (0 \leqq t \leqq 2\pi)$ is not only completely defined by the operator U, but in turn completely defines this operator.

Completing the present section we show that the operator E_t, for any fixed t, reduces the operator U, and moreover, reduces each integral power of the operator U. This is equivalent to the assertion that

$$(U^k E_t f, g) = (E_t U^k f, g) \qquad (\pm \mathrm{k} = 0, 1, 2, \ldots).$$

for arbitrary $f, g \in \mathrm{H}$, i.e., the operators U^k and E_t are permutable. The proof follows at once from the formula (3′); in fact

$$(U^k E_t f, g) = \int_0^{2\pi} e^{iks} d_s(E_s E_t f, g) = \int_0^{t} e^{iks} d(E_s f, g),$$

and, on the other hand,

$$(E_t U^k f, g) = (U^k f, E_t g) = \int_0^{2\pi} e^{iks} d_s(E_s f, E_t g) =$$

$$= \int_0^{2\pi} e^{iks} d_s(E_t E_s f, g) = \int_0^{t} e^{iks} d(E_s f, g).$$

63. Operators Represented by Stieltjes Integrals

In the preceding section we defined the integral

$$\int_0^{2\pi} e^{ikt} dE_t$$

as the limit, in the sense of strong convergence, of the sequence of operators

$$e^{ikt_1}E(\varDelta_1) + e^{ikt_2}E(\varDelta_2) + \dots + e^{ikt_n}E(\varDelta_n),$$

which have the form of Stieltjes integral sums. The present section is devoted to a certain generalization of this construction.

Let us take a particular resolution of the identity E_t in a finite or infinite interval $[\alpha, \beta]$ and a particular real or complex function $\varphi(t)$ which we assume to be continuous at every point of $[\alpha, \beta]$ with the possible exception of a finite number of points. We form the expression

$$\int_\alpha^\beta \overline{\varphi(t)}\, d\overline{(E_t f, h)},$$

and we denote by M the set (linear manifold) of all those elements f for which this expression is a linear functional of f. By the theorem of F. Riesz there exists a unique element Tf which depends on $f \in$ M, such that

$$\int_\alpha^\beta \overline{\varphi(t)}\, d\overline{(E_t f, h)} = (h, Tf),$$

or

$$\int_\alpha^\beta \varphi(t)\, d(E_t f, h) = (Tf, h). \tag{1}$$

It is easy to see that T is a linear operator with domain $\mathrm{D}_T = \mathrm{M}$. The present section is devoted to the study of this operator.

For simplicity we assume that the function $\varphi(t)$ is continuous on the half-closed interval $[\alpha, \beta)$. Thus, we will have only one exceptional point (the right end point of the interval). This assumption, however, does not decrease the generality of our conclusions, since the case of any finite number of exceptional points can be reduced to the simpler one by subdividing the interval into parts.

We proceed to show that the domain M of the operator T is the set of all vectors f for which

$$\int_\alpha^\beta |\varphi(t)|^2\, d(E_t f, f) < \infty. \tag{2}$$

The first step is to prove that the inclusion $f \in$ M implies inequality (2). Fix $f \in$ M and, in (1), let

$$h = Tf.$$

Then

$$\|Tf\|^2 = \int_\alpha^\beta \varphi(t)\,d(E_tf, Tf) = \lim_{\gamma\to\beta}\int_\alpha^\gamma \varphi(t)\,d(E_tf, Tf).$$

But

$$(E_tf, Tf) = \overline{(Tf, E_tf)} = \int_\alpha^\beta \overline{\varphi(s)}\,d_s\overline{(E_sf, E_tf)} = \int_\alpha^t \overline{\varphi(s)}\,d(E_sf, f).$$

Therefore,

$$\|Tf\|^2 = \lim_{\gamma\to\beta}\int_\alpha^\gamma |\varphi(t)|^2 d(E_tf, f) = \int_\alpha^\beta |\varphi(t)|^2 d(E_tf, f),$$

which yields inequality (2).

Let us assume now that inequality (2) holds for a fixed element f, and prove that $f \in \mathrm{M}$. Since

$$\int_{\gamma'}^{\gamma''} \varphi(t)\,d(E_tf, h) = \lim_{n\to\infty}\sum_{k=1}^{n} \varphi(t_k)\,(E(\Delta_k)f, h),$$

and

$$|\sum_{k=1}^{n} \varphi(t_k)(E(\Delta_k)f, h)| \leqq \sum_{k=1}^{n} |\varphi(t_k)| \cdot |(E(\Delta_k)f, E(\Delta_k)h| \leqq$$

$$\leqq \sqrt{\sum_{k=1}^{n} |\varphi(t_k)|^2 (E(\Delta_k)f, f)}\sqrt{\sum_{k=1}^{n} (E(\Delta_k)h, h)},$$

and

$$\sum_{k=1}^{n} (E(\Delta_k)h, h) \leqq \|h\|^2,$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} |\varphi(t_k)|^2 (E(\Delta_k)f, f) = \int_{\gamma'}^{\gamma''} |\varphi(t)|^2 d(E_tf, f),$$

we see that

$$\left|\int_{\gamma'}^{\gamma''} \varphi(t)\,d(E_tf, h)\right| \leqq \|h\| \sqrt{\int_{\gamma'}^{\gamma''} |\varphi(t)|^2 d(E_tf, f)}.$$

By this inequality, in which γ', γ'' are any numbers which satisfy the inequality $\alpha \leqq \gamma' < \gamma'' < \beta$, we conclude that the improper integral

$$\int_\alpha^\beta \varphi(t)\,d(E_tf, h)$$

exsits and also that

$$\left|\int_\alpha^\beta \varphi(t)\,d(E_tf, h)\right| \leqslant \|h\| \sqrt{\int_\alpha^\beta |\varphi(t)|^2 d(E_tf, f)}.$$

This implies that $f \in \mathrm{M}$.

From our proof it follows that if the function $\varphi(t)$ is bounded in the interval (α, β), then T is a bounded operator which is defined everywhere in H. In this case T can be defined also as the Stieltjes integral operator

$$\int_{\alpha}^{\beta} \varphi(t)\, dE_t, \tag{3}$$

i.e., as the strong limit of some operator integral sum, similar to that which was done in the preceding section.

Further, in this case it is immediately verified that the operator T^* adjoint to T has the form

$$\int_{\alpha}^{\beta} \overline{\varphi(t)}\, dE_t.$$

If the function $\varphi(t)$ is not bounded, then the domain of the operator does not coincide with the whole space H, but it is dense in H. If, for example, $\varphi(t)$ is bounded on each interval $[\alpha, \gamma]$ with $\gamma \in (\alpha, \beta)$, but is unbounded in the neighborhood of β, then D_T contains the set of all the elements in some manifold

$$(E_\gamma - E_\alpha)\,\mathrm{H}$$

with $\gamma \in (\alpha, \beta)$. This set is dense in H. From what has been said, it follows that T has an adjoint operator T^* also in case $\varphi(t)$ is not bounded. We show now that T^* is defined by the formula

$$(T^*f, g) = \int_{\alpha}^{\beta} \overline{\varphi(t)}\, d(E_t f, g),$$

where $\mathrm{D}_{T^*} = \mathrm{M} = \mathrm{D}_T$. In a similar manner, we show that in the case of an unbounded function $\varphi(t)$, the operator T can be represented by a Stieltjes integral, this time improper. For simplicity we again restrict our consideration to the case when the function $\varphi(t)$ is continuous in the half-closed interval $[\alpha, \beta)$.

First, we introduce the function

$$\varphi_\gamma(t) = \begin{cases} \varphi(t) & (\alpha \leqq t \leqq \gamma < \beta), \\ 0 & (t > \gamma), \end{cases}$$

in terms of which we define an operator T_γ as a proper Stieltjes integral

$$T_\gamma = \int_{\alpha}^{\beta} \varphi_\gamma(t)\, dE_t = \int_{\alpha}^{\gamma} \varphi(t)\, dE_t.$$

Now, T_γ is a bounded operator which is defined everywhere in H and the adjoint operator T_γ^* has the form

$$T_\gamma^* = \int_\alpha^\beta \overline{\varphi_\gamma(t)}\, dE_t = \int_\alpha^\gamma \overline{\varphi(t)}\, dE_t.$$

For any $f, h \in$ H

$$(T_\gamma f, h) = \int_\alpha^\beta \varphi_\gamma(t)\, d(E_t f, h) = \int_\alpha^\gamma \varphi(t)\, d(E_t f, h).$$

Now we prove that for any $f \in$ M the equation

$$Tf = \lim_{\gamma \to \beta} T_\gamma f$$

holds, so that the operator T can be represented as an improper integral

$$\int_\alpha^\beta \varphi(t)\, dE_t,$$

with the linear manifold M as domain. Indeed, according to what has been proved above, for any $f \in$ H

$$\| T_\gamma f \|^2 = \int_\alpha^\gamma | \varphi(t) |^2 d(E_t f, f).$$

Therefore,

$$\lim_{\gamma \to \beta} \| T_\gamma f \|$$

exists if and only if $f \in$ M, in which case

$$\lim_{\gamma \to \beta} \| T_\gamma f \| = \| Tf \|. \tag{4}$$

But on the other hand, for $f \in$ M, and each $h \in$ H

$$\lim_{\gamma \to \beta} (T_\gamma f, h) = \lim_{\gamma \to \beta} \int_\alpha^\gamma \varphi(t)\, d(E_t f, h) = (Tf, h).$$

In other words, for $\gamma \to \beta$,

$$T_\gamma f \xrightarrow{w} Tf.$$

This result and (4) imply that

$$T_\gamma f \to Tf.$$

Replacing $\varphi(t)$ by $\overline{\varphi(t)}$ in the preceding development, we find that, for each $g \in$ M,

$$\int_\alpha^\beta \overline{\varphi(t)}\, dE_t g = \lim_{\gamma \to \beta} \int_\alpha^\gamma \overline{\varphi(t)}\, dE_t g = \lim_{\gamma \to \beta} T_\gamma^* g = Sg,$$

where the limit is in the sense of strong convergence. Therefore, for $f, g \in \mathrm{M}$,

$$(Tf, g) = (f, Sg),$$

which implies that

$$\mathrm{M} \subset \mathrm{D}_{T^*}. \tag{5}$$

We assume now that for some pair of elements $h, h^* \in \mathrm{H}$

$$(Tf, h) = (f, h^*)$$

for any $f \in \mathrm{M}$. For f, we substitute the element $T_\gamma^* h$, which certainly belongs to M. Noticing that

$$(E_t T_\gamma^* h, h) = \int_a^\gamma \overline{\varphi(s)}\, d_s(E_s h, E_t h),$$

we get

$$(Tf, h) = (TT_\gamma^* h, h) = \int_a^\beta \varphi(t)\, d_t\left\{\int_a^\gamma \overline{\varphi(s)}\, d_s(E_s h, E_t h)\right\} = \int_a^\gamma |\varphi(t)|^2 d(E_t h, h) =$$
$$= \| T_\gamma^* h \|^2.$$

From the equation

$$\| T_\gamma^* h \|^2 = (T_\gamma^* h, h^*),$$

it follows that

$$\| T_\gamma^* h \| \leqq \| h^* \|.$$

This yields the inequality

$$\lim_{\gamma \to \beta} \int_a^\gamma |\varphi(t)|^2 d(E_t h, h) \leqq \| h^* \|^2 < \infty,$$

i.e., $h \in \mathrm{M}$. And so, it is proved that $\mathrm{D}_{T^*} \subset \mathrm{M}$. From inclusion (5) it follows that $\mathrm{D}_{T^*} = \mathrm{M} = \mathrm{D}_T$ and also that

$$T^* = S = \int_a^\beta \overline{\varphi(t)}\, dE_t.$$

Since the transition from T to T^* resulted from the replacement of $\varphi(t)$ by the complex conjugate function $\overline{\varphi(t)}$, we have $(T^*)^* = T$. Among other things, it follows that the operator T is closed.

There is still another way to construct the operator T in the case of an unbounded function $\varphi(t)$. In place of approximating $\varphi(t)$ by the functions $\varphi_\gamma(t)$, it is possible to use the function $\varphi(t)$ directly to define the operator

$$\int_a^\beta \varphi(t)\, dE_t$$

first for the elements, for which the integral exists as the strong limit of integral sums (in the proper sense), and then to close that operator. This closure will be our operator T.

Until now we have been considering the class **K** of all functions $\varphi(t)$ which are continuous with the possible exception of a finite number of points. A further generalization is related to the larger class $\tilde{\mathbf{K}}$, which is defined by the condition that $\varphi(t) \in \tilde{\mathbf{K}}$ if and only if there exists some $f \in \mathrm{H}$ such that $\varphi(t) \in L^2_\sigma$, where $\sigma(t) = (E_t f, f)$ $(\alpha \leqq t \leqq \beta)$.

In order to define the integral

$$\int_\alpha^\beta \varphi(t)\, dE_t f$$

for $\varphi(t) \in \tilde{\mathbf{K}}$, we choose a sequence $\{\varphi_n(t)\}_1^\infty$, of functions in **K** which converge in the sense of the metric of L^2_σ to the function $\varphi(t)$. We let

$$g_n = \int_\alpha^\beta \varphi_n(t)\, dE_t f \qquad (n = 1, 2, 3, \ldots).$$

Since, for functions of the class **K**,

$$g_m - g_n = \int_\alpha^\beta \{\varphi_m(t) - \varphi_n(t)\}\, dE_t f$$

and

$$\| g_m - g_n \|^2 = \int_\alpha^\beta | \varphi_m(t) - \varphi_n(t) |^2 d(E_t f, f)$$

the sequence $\{g_k\}_1^\infty$ converges to some element $g \in \mathrm{H}$. We define

$$\int_\alpha^\beta \varphi(t)\, dE_t f = g.$$

Denoting by D_T the set of elements $f \in \mathrm{H}$ such that $\varphi(t) \in L^2_\sigma$, where $\sigma(t) = (E_t f, f)$, we obtain an operator defined by the Lebesgue-Stieltjes integral

$$Tf = \int_\alpha^\beta \varphi(t)\, dE_t f.$$

If the function $\varphi(t)$ belongs to the class **K**, then this Lebesgue-Stieltjes integral coincides with the integral considered earlier.

64. The Integral Representation of a Group of Unitary Operators

The method which we applied in Section 62 in the study of unitary operators, we apply now in some other cases. We use it to obtain an integral representation of *a group of unitary operators* (in the present section) and the *resolvent of a self-adjoint operator* (in the following section).

Let a family of unitary operators U_s be given which depend on a parameter s $(-\infty < s < \infty)$ and which satisfy the following conditions

1) $U_sU_t = U_{s+t}$,

2) $U_0 = E$,

3) (U_tf, g) is a continuous function of t for any $f, g \in \mathrm{H}$. From 1) and 2) it follows that $U_t^{-1} = U_{-t}$. But since $U_t^* = U_t^{-1}$, we have $U_t^* = U_{-t}$.

The family of operators under consideration comprises a continuous abelian group.

We select an arbitrary element $f \in \mathrm{H}$ and consider the function

$$F(t) = (U_tf, f).$$

This function is positive definite. In fact, it is continuous and, for arbitrary real t_α and complex $\rho_\alpha (\alpha = 1, 2, 3, \ldots, n)$,

$$\sum_{\alpha,\beta=1}^{n} F(t_\alpha - t_\beta)\rho_\alpha\bar{\rho}_\beta = \sum_{\alpha,\beta=1}^{n} (U_{t_\alpha}U_{-t_\beta}f, f)\rho_\alpha\bar{\rho}_\beta =$$

$$= \sum_{\alpha,\beta=1}^{n} (\rho_\alpha U_{t_\alpha}f, \rho_\beta U_{t_\beta}f) = \|\sum_{\alpha=1}^{n} \rho_\alpha U_{t_\alpha}f\|^2 \geqq 0.$$

Therefore, we can apply the theorem of Bochner to the function $F(t)$. By this theorem, there exists a unique nondecreasing function $\omega(s) = \omega(s; f)$ such that

$$\omega(-\infty) = \lim_{s \to -\infty} \omega(s) = 0, \qquad \omega(s - 0) = \omega(s) \qquad (-\infty < s < \infty)$$

and

$$F(t) = \int_{-\infty}^{\infty} e^{ist} d\omega(s) \qquad (-\infty < t < \infty).$$

For $t = 0$ it follows that

$$(f, f) = F(0) = \int_{-\infty}^{\infty} d\omega(s) = \omega(\infty).$$

Further, the function $\omega(s; f)$ determines a function $\omega(s; f, g)$ such that

$$(U_tf, g) = \int_{-\infty}^{\infty} e^{ist} d\omega(s; f, g).$$

Under the normalization conditions

$$\left.\begin{aligned} \omega(-\infty; f, g) &= \lim_{s\to-\infty} \omega(s; f, g) = 0, \\ \omega(s-0; f, g) &= \omega(s; f, g); \end{aligned}\right\} \qquad (-\infty < s \leqq \infty),$$

this representation is unique. By this uniqueness, as before, it is proved that $\omega(s; f, g)$ is a bilinear functional of f, g, the norm of which does not exceed unity and which has the following property:

$$\overline{\omega(s; f, g)} = \omega(s; f, g).$$

Therefore, there exists a one parameter family of bounded self-adjoint operators E_s such that

$$\omega(s; f, g) = (E_s f, g) \qquad (-\infty \leqq s \leqq \infty).$$

The proof that $E_s(-\infty \leqq s \leqq \infty)$ is a resolution of this identity does not differ much from the analogous proof in Section 62. Indeed, on one hand,

$$(U_{t+\tau} f, g) = \int_{-\infty}^{\infty} e^{is(t+\tau)} d(E_s f, g) = \int_{-\infty}^{\infty} e^{ist} d_s \left\{ \int_{-\infty}^{s} e^{i\sigma\tau} d(E_\sigma f, g) \right\},$$

and, on the other

$$(U_{t+\tau} f, g) = (U_t U_\tau f, g) = \int_{-\infty}^{\infty} e^{ist} d_s \, (E_s U_\tau f, g).$$

Therefore, by the uniqueness of the representation,

$$(E_s U_\tau f, g) = \int_{-\infty}^{s} e^{i\sigma\tau} d\,(E_\sigma f, g).$$

We now make use of the representation,

$$(E_s U_\tau f, g) = (U_\tau f, E_s g) = \int_{-\infty}^{\infty} e^{i\sigma\tau} d_\sigma (E_\sigma f, E_s g).$$

Again by the uniqueness of the representation, we obtain the equation

$$(E_\sigma f, g) = (E_\sigma f, E_s g) = (E_s E_\sigma f, g) \qquad (\sigma \leqq s),$$

from which it follows that

$$E_u E_v = E_s \qquad (s = \min\{u, v\}).$$

Further, the normalization conditions satisfied by ω yield

$$E_{s-0} = E_s, \qquad E_{-\infty} = 0, \qquad E_\infty = E$$

first in the sense of weak and then in the sense of strong convergence.

In complete correspondence with the considerations of the preceding section, the integral representation

$$U_t f = \int_{-\infty}^{\infty} e^{ist} dE_s f$$

has a direct meaning as an operator in addition to being a symbolic notation for the equation

$$(U_t f, g) = \int_{-\infty}^{\infty} e^{ist} d(E_s f, g).$$

In conclusion, we leave to the reader the verification that the operator E_s reduces the operator U_t.

65. The Integral Representation of the Resolvent of a Self-Adjoint Operator

Let A be a self-adjoint operator and $R_z = (A - zE)^{-1}$ its resolvent, which we shall consider only for $\Im z \neq 0$. Fix $f \in \mathrm{H}$ and, for z in the upper half-plane, consider the function

$$(R_z f, f) = \varphi(z).$$

From the Hilbert relation

$$R_{z'} - R_z = (z' - z) R_{z'} R_z,$$

we get

$$\frac{\varphi(z') - \varphi(z)}{z' - z} = (R_{z'} R_z f, f) = (R_z R_z f, f) + (z' - z)(R_{z'} R_z R_z f, f).$$

Since the second term of the right member tends to zero as $z' \to z$,

$$\lim_{z' \to z} \frac{\varphi(z') - \varphi(z)}{z' - z} = (R_z R_z f, f).$$

Therefore, $\varphi(z)$ is a regular analytic function in the upper half-plane. We determine its imaginary part. It equals

$$\frac{\varphi(z) - \overline{\varphi(z)}}{2i} = \frac{(R_z f, f) - (f, R_z f)}{2i} = \frac{(R_z f, f) - (R_z^* f, f)}{2i}.$$

But (cf. Section 44)

$$R_z^* = R_{\bar{z}}.$$

Hence, again by Hilbert's relation,

$$(R_z f, f) - (R_z^* f, f) = (z - \bar{z})(R_{\bar{z}} R_z f, f) = (z - \bar{z})(R_z f, R_z f).$$

But, since $z = x + iy$ where $y > 0$,

$$\Im\, \varphi(z) = y (R_z f, R_z f) \geqq 0,$$

i.e., the imaginary part of the function $\varphi(z)$ is non-negative in the upper half-plane; moreover, it is positive if $f \neq 0$, since $R_z f = 0$ implies that $f = 0$.

We prove also that

$$\sup_{y>0} y \,|\, \varphi(iy) \,| < \infty.$$

Indeed, we saw in Section 43 that

$$\| R_z f \| \leqq \frac{1}{y} \| f \|.$$

Therefore,

$$y \,|\, \varphi(iy) \,| = y \,|\, (R_{iy} f, f) \,| \leqq (f, f).$$

We now recall Theorem 3 of Section 59. By this theorem there exists a unique nondecreasing function of bounded variation $\omega(t) = \omega(t; f)$, which satisfies the normalization conditions

$$\omega(-\infty) = \lim_{t \to -\infty} \omega(t) = 0, \quad \omega(t-0) = \omega(t) \quad (-\infty < t \leqq \infty),$$

and the equation

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{t - z} \quad (\Im z > 0).$$

Thus, it is proved that for $\Im z > 0$

$$(1) \qquad (R_z f, f) = \int_{-\infty}^{\infty} \frac{d\omega(t; f)}{t - z}.$$

From this representation it follows that

$$(f, R_{\bar{z}} f) = \int_{-\infty}^{\infty} \frac{d\omega(t; f)}{t - z},$$

and, hence,

$$(2) \qquad (R_{\bar{z}} f, f) = \int_{-\infty}^{\infty} \frac{d\omega(t; f)}{t - \bar{z}}.$$

If z lies in the upper half-plane, then $\bar{z}$ lies in the lower. Therefore, from formulas (1) and (2) it follows that the representation

$$(R_z h, h) = \int_{-\infty}^{\infty} \frac{d\omega(t; h)}{t - z}$$

is valid for each nonreal z and each $h \in \mathrm{H}$. We define

$$\omega(t; f, g) = \frac{1}{4} \omega(t; f+g) - \frac{1}{4} \omega(t; f-g) + \frac{i}{4} \omega(t; f+ig) - \frac{i}{4} \omega(t; f-ig),$$

and we find that, for each nonreal z and every $f, g \in \mathrm{H}$,

$$(3) \qquad (R_z f, g) = \int_{-\infty}^{\infty} \frac{d\omega(t; f, g)}{t - z}.$$

Here $\omega(t; f, g)$ is a complex-valued function of bounded variation equal to zero for $t = -\infty$ and continuous from the left at each point $t > -\infty$. It is not difficult to verify that under these normalization conditions, the integral representation (3) is unique. In fact, otherwise there would exist a complex-valued function of bounded variation

$$\sigma(t) = \alpha(t) + i\beta(t)$$

for which

$$(4) \qquad \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - \bar{z}} = 0$$

for every nonreal z. Then, by (4),

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - \bar{z}} = 0,$$

or

$$(4') \qquad \int_{-\infty}^{\infty} \frac{d\overline{\sigma(t)}}{t - z} = 0.$$

A comparison of (4) with (4′) shows that for every nonreal z

$$\int_{-\infty}^{\infty} \frac{d\alpha(t)}{t - z} = \int_{-\infty}^{\infty} \frac{d\beta(t)}{t - z} = 0.$$

But from theorem (3) of Section 59 and the normalization conditions it follows that

$$\alpha(t) = \beta(t) = 0.$$

Now that the uniqueness of (3) is established, it is easy to prove that

$$(5) \qquad \overline{\omega(t; f, g)} = \omega(t; g, f),$$

and

$$\omega(t; \alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \omega(t; f_1, g) + \alpha_2 \omega(t; f_2, g).$$

We show next that

$$\omega(t; f, f) \leqq (f, f).$$

Then it will follow from the theorem in Section 20 that $\omega(t; f, g)$ is a bilinear functional of f, g with norm not exceeding unity. We already noted that

$$y \,|(R_{iy} f, f)|\, \leqq (f, f) \qquad (y > 0),$$

or, equivalently,

$$\left|\int_{-\infty}^{\infty}\frac{y\,d\omega(t;f,f)}{t-iy}\right| \leqq (f,f).$$

We split the integral into three parts and find that

$$\left|\int_{-A}^{A}\frac{y\,d\omega(t;f,f)}{t-iy}\right| \leqq (f,f)+\int_{-\infty}^{-A}d\omega(t;f,f)+\int_{A}^{\infty}d\omega(t;f,f).$$

Let $y \to \infty$ to obtain

$$\int_{-A}^{A}d\omega(t;f,f) \leqq (f,f)+\int_{-\infty}^{-A}d\omega(t;f,f)+\int_{A}^{\infty}d\omega(t;f,f).$$

Now let $A \to \infty$ to get

$$\omega(\infty;f,f) \leqq (f,f).$$

Since $\omega(t;f,g)$ is a nondecreasing function of t, the inequality

$$\omega(t;f,f) \leqq (f,f)$$

is proved.

By the theorem on the general form of a bilinear functional, there exists a family of operators $E_t(-\infty \leqq t \leqq \infty)$ such that

$$\omega(t;f,g) = (E_t f, g).$$

From the relation (5) it follows that

$$(E_t f, g) = (f, E_t g),$$

i.e., the operator E_t is self-adjoint. Further, the function $(E_t f, f)$ is a nondecreasing function of t.

These considerations yield the equation

$$(R_z f, g) = \int_{-\infty}^{\infty}\frac{1}{t-z}\,d(E_t f, g),$$

which is valid for arbitrary $f, g \in \mathrm{H}$ and each nonreal z. From this equation it follows that

$$(R_z f, R_{z'} g) = \int_{-\infty}^{\infty}\frac{1}{t-z}\,d(E_t f, R_{\bar{z}'} g), \tag{6}$$

and, on the other hand,

$$(R_z f, R_{\bar{z}'} g) = (R_{z'} R_z f, g) = \frac{1}{z'-z}\{(R_{z'} f, g) - (R_z f, g)\} =$$

$$= \frac{1}{z'-z}\int_{-\infty}^{\infty}\left(\frac{1}{t-z'}-\frac{1}{t-z}\right)d(E_t f, g) = \int_{-\infty}^{\infty}\frac{1}{(t-z)(t-z')}\,d(E_t f, g).$$

And so, besides representation (6), we also have the representation

$$(R_z f, R_{\bar{z}'} g) = \int_{-\infty}^{\infty} \frac{1}{t-z} d_t \left\{ \int_{-\infty}^{t} \frac{1}{s-z'} d(E_s f, g) \right\}.$$

These representations must agree. Therefore,

$$\int_{-\infty}^{t} \frac{1}{s-z'} d(E_s f, g) = (E_t f, R_{\bar{z}'} g) = (R_{\bar{z}'} E_t f, g),$$

or

$$\int_{-\infty}^{t} \frac{1}{s-z'} d(E_s f, g) = \int_{-\infty}^{\infty} \frac{1}{s-z'} d_s(E_s E_t f, g).$$

Since these representations are identical,

$$(E_s f, g) = (E_s E_t f, g) \qquad (s \leqq t).$$

It follows that

$$E_u E_v = E_s \qquad (s = \min \{u, v\}),$$

which implies that

$$E_s \leqq E_t \qquad (s \leqq t).$$

From the normalization conditions of the function $\omega(t; f, g)$ it follows immediately that

$$E_{-\infty} = 0, \quad E_{t-0} = E_t$$

first in the sense of weak, and second in the sense of strong convergence.

Furthermore, the strong limit

$$\lim_{t \to \infty} E_t = E_\infty$$

exists. We prove now that

$$E_\infty = E,$$

from which it will follow that E_t is a resolution of the identity. Let

$$E - E_\infty = F.$$

Then

$$FE_t = (E - E_\infty) E_t = E_t - \lim_{s \to \infty} E_s E_t = E_t - E_t = 0.$$

Hence, for every $f, g \in \mathrm{H}$,

$$(R_z F f, g) = \int_{-\infty}^{\infty} \frac{1}{t-z} d(E_t F f, g) = 0.$$

This implies that, for every $f \in \mathrm{H}$,

$$R_z F f = 0.$$

Therefore,

$$Ff = 0.$$

Hence, $F = 0$, which was to be shown.

The representation derived for the resolvent R_z of a self-adjoint operator A is also an illustration of the general representation of Section 63, which refers to operator integrals. This representation can be written in the form

$$R_z = \int_{-\infty}^{\infty} \frac{1}{t-z}\, dE_t \qquad (\Im z \neq 0).$$

We propose that the reader verify that each of the operators E_t reduces the resolvent R_z, where z is an arbitrary nonreal number, i.e.,

$$E_t R_z = R_z E_t.$$

66. The Integral Representation of Self-Adjoint Operators

From the general considerations of Section 63, we obtain, as a particular case, the following theorem.

THEOREM 1: *To each resolution of the identity* E_t $(-\infty \leqq t \leqq \infty)$ *there corresponds a uniquely defined self-adjoint operator*

$$B = \int_{-\infty}^{\infty} t\, dE_t. \tag{1}$$

The domain D_B *of this operator is the set of all vectors* f *for which the inequality*

$$\int_{-\infty}^{\infty} t^2 d(E_t f, f) < \infty$$

is satisfied. The left member of this inequality is equal to $\| Bf \|^2$.

The basic problem of the present section is the proof of the converse proposition. For the purposes of this proof, let A denote a given self-adjoint operator and let R_z denote its resolvent. According to the theorem of the preceding section there is a resolution of the identity which corresponds to A. We shall prove that this resolution of the identity belongs to the operator A in the sense that it generates A in the way indicated by Theorem 1. The first step of this proof is the following lemma.

LEMMA: *If* E_t $(-\infty \leqq t \leqq \infty)$ *is a resolution of the identity which belongs to the resolvent* R_z *of a self-adjoint operator* A, *then the set of all vectors* $f \in \mathrm{H}$, *for which inequality* (2) *holds coincides with the domain* D_A *of the operator* A.

Proof: We know that the vector $f = R_z h$ runs through D_A, when the vector h runs through H and z is any fixed nonreal number. We take $z=i$ and, for brevity, write

$$R_i = R, \ R_{-i} = R^*.$$

Our proposition will be proved if we verify the equivalence of the following two assertions:

α) The element f has the property

$$\int_{-\infty}^{\infty} t^2 d(E_t f, f) < \infty.$$

β) There exists a vector h such that

$$f = Rh.$$

We prove first that β implies α. Therefore, let

$$f = Rh.$$

Then

$$(E_t f, f) = (E_t Rh, Rh) = (RE_t h, Rh) = (R^*RE_t h, h) =$$
$$= \frac{1}{2i}(\{R - R^*\} E_t h, h) = \frac{1}{2i} \int_{-\infty}^{\infty} \left(\frac{1}{s-i} - \frac{1}{s+i}\right) d_s(E_s E_t h, h) =$$
$$= \int_{-\infty}^{t} \frac{1}{1+s^2} d(E_s h, h),$$

and, hence,

$$\int_{-M}^{M} t^2 d(E_t f, f) = \int_{-M}^{M} \frac{t^2}{1+t^2} d(E_t h, h) \leqq \int_{-M}^{M} d(E_t h, h) \leqq (h, h).$$

This inequality shows that

$$\int_{-\infty}^{\infty} t^2 d(E_t f, f) \leqq (h, h) < \infty,$$

so that assertion α is proved.

Now we prove that α implies β. Assume α. This implies that the vector f belongs to the domain of the operator B which was defined in Theorem 1. Let

$$h = (B - iE)f.$$

Assertion β) will be proved if we verify that

$$R(B - iE)f = f.$$

For an arbitrary vector $g \in$ H, we have the equations

$$(R(B-iE)f,g)=\int_{-\infty}^{\infty}\frac{1}{t-i}\,d(E_t(B-iE)f,g),$$

and

$$((B-iE)f,E_tg)=\int_{-\infty}^{\infty}(s-i)d_s(E_sf,E_tg)=\int_{-\infty}^{t}(s-i)d(E_sf,g).$$

Therefore,

$$(R(B-iE)f,g)=\int_{-\infty}^{\infty}\frac{t-i}{t-i}\,d(E_tf,g)=(f,g).$$

Since g is arbitrary,

(3) $$R(B-iE)f=f.$$

THEOREM 2: *The resolution of the identity* $E_t(-\infty\leqq t\leqq\infty)$ *determined by the resolvent* R_z *of a self-adjoint operator* A *is the resolution of the identity which belongs to the operator* A *in the sense that, first,* D_A *is the set of all vectors* f *for which*

$$\int_{-\infty}^{\infty}t^2d(E_tf,f)<\infty,$$

and, second, for any $f\in\mathrm{D}_A$

$$Af=\int_{-\infty}^{\infty}t\,dE_tf.$$

Proof: We construct the operator B, which corresponds, in the sense of Theorem 1, to the resolution of the identity determined by the resolvent R_z of the operator A. By the lemma, $\mathrm{D}_B=\mathrm{D}_A$. Hence, it remains to prove that the equation

(4) $$Bf=Af$$

holds for each $f\in\mathrm{D}_A$. But, in Section 44, it was shown that the equation,

$$R_zg=0,$$

with z nonreal, is satisfied only for $g=0$. Therefore, instead of (4), it is sufficient to prove that

$$RBf=RAf,$$

or

$$R(B-iE)f=R(A-iE)f.$$

This equation holds since the left side equals f by (3) and the right side equals f by the definition of the resolvent.

THEOREM 3: *The resolution of the identity* $E_t(-\infty\leqq t\leqq\infty)$ *belongs to the self-adjoint operator* A *if and only if*

1° E (Δ) *reduces the operator* A *for any interval* $\Delta \subset [-\infty, \infty]$,

2° *the relation*

$$f \in (E_t - E_s)\,\mathrm{H} \qquad (-\infty \leqq s < t \leqq \infty)$$

implies the inequality

$$s\,\|f\|^2 \leqq (Af, f) \leqq t\,\|f\|^2.$$

Proof: The necessity of condition 1° was noted earlier. The necessity of condition 2° follows from the representation

$$(Af, f) = \int_s^t \tau d(E_\tau f, f),$$

which is valid for $f \in (E_t - E_s)$ H.

We turn now to the proof of the sufficiency of the conditions. Therefore, assume conditions 1° and 2° and choose an arbitrary element $f \in \mathrm{D}_A$. By condition 1°, $(E_\beta - E_\alpha)f$ belongs to the domain D_A, for any α and β. Supposing that α and β are finite, we form the partition

$$\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n = \beta \tag{5}$$

of the interval $[\alpha, \beta]$, and we represent $(E_\beta - E_\alpha)f$ in the form

$$(E_\beta - E_\alpha)f = \sum_{k=0}^{n-1} (E_{\alpha_{k+1}} - E_{\alpha_k})f.$$

By 1°,

$$(E_\beta - E_\alpha)\,Af = \sum_{k=0}^{n-1} A\,(E_{\alpha_{k+1}} - E_{\alpha_k})f. \tag{6}$$

It follows from condition 2° that, for $f \in (E_t - E_s)$ H,

$$-\frac{t-s}{2}(f, f) \leqq \left(Af - \frac{s+t}{2}f, f\right) \leqq \frac{t-s}{2}(f, f).$$

In other words, the restriction of the self-adjoint operator

$$A - \frac{s+t}{2}E$$

to the subspace $(E_t - E_s)$ H has norm $\leqq \frac{t-s}{2}$. Therefore, we write (6) in the form

$$(E_\beta - E_\alpha)Af = \sum_{k=0}^{n-1} \frac{\alpha_k + \alpha_{k+1}}{2}(E_{\alpha_{k+1}} - E_{\alpha_k})f + \tag{7}$$
$$+ \sum_{k=0}^{n-1} \left(A - \frac{\alpha_k + \alpha_{k+1}}{2}E\right)(E_{\alpha_{k+1}} - E_{\alpha_k})f,$$

and note that

$$\left\| \sum_{k=0}^{n-1} \left(A - \frac{a_k + a_{k+1}}{2} E\right) (E_{a_{k+1}} - E_{a_k})f \right\|^2 \leqq$$

$$\leqq \sum_{k=0}^{n-1} \left(\frac{a_{k+1} - a_k}{2}\right)^2 \| (E_{a_{k+1}} - E_{a_k})f \|^2 \leqq \varepsilon^2 \|f\|^2,$$

where

$$\varepsilon = \max_k \frac{a_{k+1} - a_k}{2}.$$

In (7), let the diameter 2ε of the subdivision (5), tend to zero. This passage to the limit yields

$$(E_\beta - E_\alpha)Af = \int_\alpha^\beta t dE_t f,$$

and

$$\| (E_\beta - E_\alpha)Af \|^2 = \int_\alpha^\beta t^2 d(E_t f, f).$$

The second of these relations shows that

$$\int_{-\infty}^{\infty} t^2 d(E_t f, f) = \| Af \|^2 < \infty,$$

and the first gives

$$Af = \int_{-\infty}^{\infty} t dE_t f.$$

Therefore, the theorem is proved.

In the derivation of the integral representation of the resolvent of a self-adjoint operator (Section 65) we used only the fact that $R_z(\Im z \neq 0)$ is a family of operators, defined everywhere in H, which satisfy the following conditions:

1° $\| R_z f \| \leqq \frac{1}{|y|} \|f\| \quad (y = \Im z).$

2° $R_z^* = R_{\bar{z}}.$

3° $R_{z'} - R_z = (z' - z) R_{z'} R_z.$

4° If $R_z f = 0$ for any z, then $f = 0$.

Now we can assert that every family of operators, which satisfy these conditions is the resolvent of some self-adjoint operator.[8] Indeed, having such a family of operators, we can find its integral representation by some resolution of the identity E_t, and then construct a self-adjoint operator A with this resolution of the identity. The resolvent of this operator A is the family R_z.

Completing the present section we mention several simple facts about self-adjoint operators which are immediate consequences of the integral representation of these operators.

1° If A is a self-adjoint operator for which

$$\inf_{f \in D_A} \frac{(Af,f)}{(f,f)} = \alpha,$$
$$\sup_{f \in D_A} \frac{(Af,f)}{(f,f)} = \beta,$$

then the integral representation of A has the form

$$A = \int_{\alpha}^{\beta} t dE_t,$$

i.e.,

$$E_t = 0 \qquad (\text{for } t \leqq \alpha),$$
$$E_t = E \qquad (\text{for } t \geqq \beta).$$

2° In order that a vector $f \in$ H admit the n-fold application of the self-adjoint operator A, i.e., in order that

$$A^k f = A\,(A^{k-1} f) \qquad (k = 1, 2, \ldots, n)$$

have meaning, it is necessary and sufficient that the inequality

$$\int_{-\infty}^{\infty} t^{2n} d(E_t f, f) < \infty$$

be satisfied. If this inequality is satisfied, then for $k = 1, 2, \ldots, n$

$$A^k f = \int_{-\infty}^{\infty} t^k dE_t f, \quad \| A^k f \|^2 = \int_{-\infty}^{\infty} t^{2k} d(E_t f, f).$$

3° There exists a linear manifold dense in H which is in the domain of every positive integral power of the self-adjoint operator A. One such manifold is the set of vectors of the form $E(\Delta)\, h$, where h runs through H, and Δ runs through the set of all finite intervals on the real axis.

The proof of these assertions we leave to the reader.

---

[8] This fact can be established without the use of the integral representation of the family of operators R_z. If one uses only properties 2°, 3°, 4°, then it is easy to show that the operator A defined by the equation $A = zE + R_z^{-1}$ does not depend on z, is self-adjoint, and its resolvent is R_z. Thus, it is revealed that property 1° is a corollary which need not be assumed.

67. The Cayley Transform

In the present section we give another construction of the integral representation of a self-adjoint operator—a construction which is based on the so-called *Cayley transform.*

Let A be a closed symmetric operator and let z be a nonreal number. Let h run through D_A and let

$$(A - \bar{z}E)h = f, \tag{1}$$

$$(A - zE)h = g. \tag{2}$$

The vector f will run through some linear manifold F, and the vector g through some linear manifold G. We prove that F and G are closed and, hence, are subspaces of H. With this aim, we assume that

$$f_n = (A - \bar{z}E)h_n \qquad (n = 1, 2, 3, \ldots),$$

and

$$f_n \to f. \tag{3}$$

If $z = x + iy$, where $y \neq 0$, then

$$f_n = (A - xE)h_n - iyh_n,$$

which yields

$$\|f_n - f_m\|^2 = \|(A - xE)(h_n - h_m)\|^2 + y^2\|h_n - h_m\|^2.$$

Therefore, from (3), it follows that

$$\lim_{n\to\infty} (A - xE)h_n \text{ and } \lim_{n\to\infty} h_n$$

exist. We denote the second of these limits by h and use the fact that A is a closed operator to conclude that

$$f = (A - xE)h + iyh = (A - \bar{z}E)h.$$

Hence, $f \in$ F, so that the manifold F is closed. Analogously, it is proved that G is closed.

By means of relations (1) and (2), the manifold D_A is mapped in one-to-one manners onto F and G, respectively. Therefore, for each $f \in$ F there is one and only one element $h \in D_A$ which satisfies relation (1). Having found this element h, we determine the element g by formula (2). Thus, to each element $f \in$ F is associated a unique element $g \in$ G, i.e., we have an operator V with domain $D_V =$ F and range $\Delta_V =$ G:

$$g = Vf.$$

It is easy to see that V is an isometric operator. Indeed, it is linear and

$$\|f\|^2 = \|(A - xE)h\|^2 + y^2\|h\|^2,$$

$$\|g\|^2 = \|(A - xE)h\|^2 + y^2\|h\|^2.$$

The isometric operator V is called the Cayley transform of the symmetric operator A.

Let $z = i$. Then equations (1) and (2) have the form

$$(1') \qquad (A + iE)h = f,$$

$$(2') \qquad (A - iE)h = g = Vf.$$

From these relations it follows that

$$h = \frac{1}{2i}(f - g) = \frac{1}{2i}(E - V)f,$$

$$Ah = \frac{1}{2}(f + g) = \frac{1}{2}(E + V)f.$$

Therefore,

$$(4) \qquad Ah = i(E + V)(E - V)^{-1}h.$$

Thus, the symmetric operator A is expressed in terms of its Cayley transform V.

Especially important is the case when V is a unitary operator.

THEOREM: *Let the Cayley transform of a symmetric operator A be a unitary operator U with a resolution of the identity $F_s (0 \leqq s \leqq 2\pi)$. If*

$$E_t = F_s \qquad \left(t = -\cot\frac{s}{2}\right),$$

then the domain D_A of A is the set of all vectors h for which

$$\int_{-\infty}^{\infty} t^2 d(E_t h, h) < \infty.$$

The operator A has the integral representation

$$Ah = \int_{-\infty}^{\infty} t dE_t h.$$

Proof: Let $h \in D_A$ and f be the element which corresponds to h by the mapping (1′). Then

$$h = \frac{1}{2i}(E - U)f.$$

In this case,

$$(5) \qquad F_s h = \frac{1}{2i}(E - U)F_s f = \frac{1}{2i}\int_0^{2\pi}(1 - e^{i\tau})d_\tau(F_\tau F_s f) =$$

$$= \frac{1}{2i}\int_0^{s}(1 - e^{i\tau})d F_\tau f,$$

and

$$
\begin{aligned}
(F_s h, h) &= \frac{1}{4}(\{E-U\}F_s f, \{E-U\}f) = \\
&= \frac{1}{4}(\{E-U^{-1}\}\ \{E-U\}F_s f, f) = \\
&= \frac{1}{4}(\{2E-U-U^{-1}\}F_s f, f) = \qquad (6)\\
&= \frac{1}{4}\int_0^{2\pi}(2-e^{i\tau}-e^{-i\tau})\,d_\tau(F_\tau F_s f, f) = \\
&= \int_0^{s}\sin^2\frac{\tau}{2}\,d(F_\tau f, f).
\end{aligned}
$$

On the other hand

$$
Ah = \frac{1}{2}(E+U)f = \frac{1}{2}\int_0^{2\pi}(1+e^{is})\,dF_s f, \tag{7}
$$

and

$$
\begin{aligned}
\|Ah\|^2 = \frac{1}{4}(\{E+U\}f, \{E+U\}f) &= \frac{1}{4}(\{2E+U+U^{-1}\}f, f) = \\
&= \int_0^{2\pi}\cos^2\frac{s}{2}\,d(F_s f, f). \qquad (8)
\end{aligned}
$$

A comparison of (6) and (8) reveals that

$$
\|Ah\|^2 = \int_0^{2\pi}\cot^2\frac{s}{2}\sin^2\frac{s}{2}\,d(F_s f, f) = \int_0^{2\pi}\cot^2\frac{s}{2}\,d(F_s h, h).
$$

In the same manner it follows from (5) and (7) that, for each $h' \in \mathrm{H}$,

$$
(Ah, h') = -\int_0^{2\pi}\cot\frac{s}{2}\,d(F_s h, h').
$$

We now let

$$
t = -\cot\frac{s}{2}, \quad E_t = F_s.
$$

Then E_t is a resolution of the identity in the interval $[-\infty, \infty]$. If $h \in \mathrm{D}_A$, then

$$
\int_{-\infty}^{\infty} t^2 d(E_t h, h) < \infty \tag{9}
$$

and

$$(10) \qquad Ah = \int_{-\infty}^{\infty} t\, dE_t h.$$

In order to complete the proof of the theorem, it remains to prove that if

$$(11) \qquad \int_{-\infty}^{\infty} t^2 d(E_t h, h) = \int_0^{2\pi} \cot^2 \frac{s}{2} d(F_s h, h) < \infty,$$

then $h \in D_A$. Therefore, assume that (11) holds with a particular element h. Then the operator

$$-\int_0^{2\pi} e^{-is/2} \frac{1}{\sin \frac{s}{2}} dF_s$$

is defined for the element h. Let

$$-\int_0^{2\pi} e^{-is/2} \frac{1}{\sin \frac{s}{2}} dF_s h = f.$$

Then, for each $h' \in H$,

$$-\int_0^{2\pi} e^{-is/2} \frac{1}{\sin \frac{s}{2}} d(F_s h, h') = (f, h').$$

Setting

$$h' = -\frac{1}{2i}(E - U^{-1})f',$$

we find that

$$\left(\frac{1}{2i}\{E - U\}f, f'\right) = (f, h') = -\frac{1}{2i}\int_0^{2\pi} \frac{e^{-is/2}}{\sin \frac{s}{2}} d(F_s h, \{E - U^{-1}\}f') =$$

$$= -\frac{1}{2i}\int_0^{2\pi} \frac{e^{-is/2}}{\sin \frac{s}{2}} (1 - e^{is}) d(F_s h, f') = \int_0^{2\pi} d(F_s h, f') = (h, f').$$

Since f' is arbitrary this implies that

$$h = \frac{1}{2i}(E - U)f.$$

In other words, the element h has a representation, which implies that it belongs to D_A.

Since the Cayley transform of a self-adjoint operator is a unitary operator, it follows from the theorem that every self-adjoint operator has an integral representation, which fact we also obtained in the preceding section.

But the proof of the theorem yields another fact: If the Cayley transform of a symmetric operator A is a unitary operator, then A is a self-adjoint operator. Indeed, from the theorem it follows that the operator A has the representation (10) and is defined for all the elements h for which inequality (9) holds. By Theorem 1 of the preceding section, such an operator is necessarily self-adjoint.

68. The Spectra of Self-Adjoint and Unitary Operators

Let E_t be a resolution of the identity. Without loss of generality we suppose that E_t is defined for $-\infty \leqq t \leqq \infty$. Indeed, if E_t is defined originally for $t \in [\alpha, \beta]$, then we can extend the definition by letting $E_t = E$ for $t \geqq \beta$ and $E_t = 0$ for $t \leqq \alpha$.

We call the point t a *point of constancy* of E_t if there exists an $\epsilon > 0$ such that

$$E_{t+\varepsilon} - E_{t-\varepsilon} = 0,$$

and a *point of growth* otherwise. Furthermore, we call the point t a *jump point* if

$$E_{t+0} - E_t \neq 0,$$

and a *continuity point* if

$$E_{t+0} - E_t = 0.$$

Continuity points which are also points of growth we call *points of continuous growth.*

Let E_t be the resolution of the identity of a self-adjoint operator A. We prove that

(a) A real number λ is a regular point of A if and only if λ is a point of constancy of E_t.

(b) A real number λ is an eigenvalue of A if and only if λ is a jump point of E_t.

In order to prove assertion (a) we use the equation

$$(1) \qquad \|(A - \lambda E)f\|^2 = \int_{-\infty}^{\infty} (t - \lambda)^2 \, d(E_t f, f),$$

which is valid for $f \in D_A$. If λ is a point of constancy of E_t, then the function $(E_t f, f)$ is constant in some neighborhood of the point λ. Let ϵ be the radius of this neighborhood. Equation (1) yields the inequality

$$\|(A - \lambda E)f\|^2 \geqq \int_{-\infty}^{\lambda-\epsilon} + \int_{\lambda+\epsilon}^{\infty} (t - \lambda)^2 d(E_t f,f) \geqq \varepsilon^2(f,f),$$

from which it follows that λ is a regular point of A.

Conversely, if λ is a regular point, then, for some $\epsilon > 0$ and every $f \in D_A$,

$$\|(A - \lambda E)f\| \geqq \epsilon \|f\|,$$

and this implies that, for $f \in D_A$,

$$(2) \qquad \int_{-\infty}^{\infty} (t - \lambda)^2 d(E_t f,f) \geqq \epsilon^2 \int_{-\infty}^{\infty} d(E_t f,f).$$

Let us assume that λ is not a point of constancy of E_t. Then there exists an element g and a positive $\eta < \epsilon$ such that

$$(E_{\lambda+\eta} - E_{\lambda-\eta})g \neq 0.$$

Applying inequality (2) to the element

$$f = (E_{\lambda+\eta} - E_{\lambda-\eta})g,$$

which is known to belong to D_A, we get

$$\int_{\lambda-\eta}^{\lambda+\eta} (t - \lambda)^2 d(E_t g, g) \geqq \epsilon^2 \int_{\lambda-\eta}^{\lambda+\eta} d(E_t g, g).$$

Since this is obviously false, assertion a) is proved.

We proceed now to the proof of assertion b). Let λ be an eigenvalue of the operator A and let f be an eigenvector associated with it. Then

$$0 = \|(A - \lambda E)f\|^2 = \int_{-\infty}^{\infty} (t - \lambda)^2 d(E_t f,f).$$

This equation shows that the only point of growth of the function $(E_t f,f)$ can be the point $t = \lambda$. Since

$$0 \neq (f,f) = \int_{-\infty}^{\infty} d(E_t f,f),$$

the point $t = \lambda$ is definitely a point of growth. But an isolated point of growth is a jump point. Therefore

$$(E_{\lambda+0} f,f) \neq (E_\lambda f,f),$$

and, consequently,

$$E_{\lambda+0} \neq E_\lambda.$$

Conversely, let λ be a point such that

$$E_{\lambda+0} \neq E_\lambda.$$

This implies that for some vector g

$$(E_{\lambda+0} - E_\lambda)g = f \neq 0.$$

The vector f belongs to D_A and

$$\| (A - \lambda E) f \|^2 = \int_{-\infty}^{\infty} (t - \lambda)^2 d(E_t f, f).$$

But the function

$$(E_t f, f) = (E_t\{E_{\lambda+0} - E_\lambda\}g, f)$$

is equal to zero for $t < \lambda$ and does not depend on t for $t > \lambda$. Hence,

$$\| (A - \lambda E) f \| = 0,$$

i.e., λ is an eigenvalue of the operator A.

From the propositions (a) and (b) it follows that each point of continuous growth of the resolution of the identity of a self-adjoint operator belongs to its continuous spectrum.

A trivial consequence of our considerations is that the spectrum of each self-adjoint operator is nonvoid.

Turning now from self-adjoint operators to unitary operators we recall that every eigenvalue of a unitary operator has absolute value unity, i.e., each has the form $e^{i\lambda}$ where λ is called the *eigenfrequency* of the unitary operator. The equation

$$U^k f = e^{ik\lambda} f$$

holds for each integer k if f is an eigenvector associated with the eigenfrequency λ. Let U_s $(-\infty < s < \infty)$ be a continuous group of unitary operators. In this case λ is called an eigenfrequency of the group of operators if an element $f \neq 0$ (an eigenvector) exists such that, for every real s,

$$U_s f = e^{is\lambda} f$$

We now prove that: *a real number λ is an eigenfrequency of a unitary operator U (respectively, a continuous group of unitary operators U_s) if and only if λ is a jump point of the corresponding resolution of the identity.*

For definiteness, we consider the case of a group of unitary operators. Let λ be an eigenfrequency of this group and let f be an associated eigenvector. Then, for each real s,

$$U_s f - e^{is\lambda} f = 0.$$

But since

$$\|(U_s - e^{is\lambda}E)f\|^2 = (\{U_{-s} - e^{-is\lambda}E\}\{U_s - e^{is\lambda}E\}f,f) =$$
$$= (\{2E - e^{is\lambda}U_{-s} - e^{-is\lambda}U_s\}f,f) =$$
$$= \int_{-\infty}^{\infty}(2 - e^{is(\lambda-t)} - e^{-is(\lambda-t)})\, d(E_t f,f) =$$
$$= \int_{-\infty}^{\infty} 4\sin^2\frac{s(\lambda-t)}{2}\, d(E_t f,f),$$

the only possible point of growth of the function $(E_t f,f)$ is $t = \lambda$. This must be a point of growth, since otherwise the relation

$$0 \neq (f,f) = \int_{-\infty}^{\infty} d(E_t f,f)$$

would not be possible. Hence,

$$(E_{\lambda+0}f,f) \neq (E_\lambda f,f)$$

and

$$E_{\lambda+0} \neq E_\lambda.$$

We remark that the role of the non-negative factor $(\lambda - t)^2$ in the case of the self-adjoint operator is played now by $4\sin^2\frac{s(\lambda-t)}{2}$. Other than this, there is no difference in the proof. We refer to the second part of the assertion and also to the following proposition: the point λ is a point of constancy of the resolution of the identity of a unitary operator U (respectively, a group of unitary operators U_s) if and only if $e^{ik\lambda}$ for any integer k (respectively, $e^{is\lambda}$ for any real s) is a regular point of the operator U^k (respectively, of the operator U_s).

Since the resolution of the identity E_t of a self-adjoint or unitary operator completely determines the spectrum of the operator, one often calls E_t the *spectral function*.

Whether or not a given point λ belongs to the spectrum of a self-adjoint operator A is determined by the behavior of its resolvent R_z for z in the neighborhood of the point λ. The circumstances are contained in the proposition: *a real point λ does not belong to the spectrum of a self-adjoint operator A, and hence, is a regular point of this operator if and only if, for each fixed element $f \in \mathrm{H}$, the function $(R_z f,f)$*

1) *is regular for z in some neighborhood of the point* λ, *and*

2) *takes on real values in some interval* $\lambda - \delta < t < \lambda + \delta$.

The necessity of these conditions follows immediately from the integral representation of the resolvent and assertion a) of the present section. The sufficiency follows from the Stieltjes inversion formula (cf. Section 59) and assertion a) of the present section.

In the following section we shall become acquainted with a class of operators for which the above conditions need to be verified only for one (specially chosen) element f.

69. The Simple Spectrum

In linear algebra and the elementary theory of integral equations, the spectrum of an operator is called *simple*, if the multiplicity of each eigenvalue of this operator equals one. This definition does not apply to arbitrary operators in Hilbert space, since in general the set of eigenvalues of an operator does not exhaust its spectrum. We make the following definition.

DEFINITION: *The spectrum of a self-adjoint (respectively, unitary) operator is called simple if there exists a vector* $g \in \mathrm{H}$ *(generating vector) such that the linear envelope of the set of vectors* $E(\Delta)g$, *where* Δ *runs through the set of all intervals on the real axis (respectively, the segment* $[0, 2\pi]$) *is dense in* H.

By this definition the spectra of a self-adjoint operator and of its Cayley transform are simultaneously simple or not simple. In addition, from the given definition it follows that in a nonseparable space a self-adjoint (unitary) operator cannot have a simple spectrum.

For operators, which occur in linear algebra and the elementary theory of integral equations, it is characteristic that the linear envelope of the set of all eigenvectors of these operators coincides with the space, or in any case is dense in it. It is not difficult to show that for an arbitrary self-adjoint (or unitary) operator in H, which possesses these properties, the new definition adopted above for the simplicity of the spectrum is in complete accord with the elementary definition mentioned at the beginning of the present section. This fact can be regarded as a justification of the general definition.

We consider as an example the multiplication operator[9] Q in L^2_σ $(-\infty, \infty)$ (cf. Section 48). The spectral function E_t of this operator is defined by the equation

$$E(\Delta)f(t) = \chi_\Delta(t)f(t) \qquad (f(t) \in L^2_\sigma),$$

---

[9] For simplicity we write Q instead of Q_σ.

where $\chi_\Delta(t)$ is the characteristic function of the interval Δ. In order to ascertain this it is sufficient to recall Theorem 3 of Section 66. The operator Q has a simple spectrum. Indeed, let us take a step function $g(t)$ which satisfies the conditions

$$g(t) = a_k > 0 \qquad (k - 1 < t \leqq k),$$

$$\sum_{k=-\infty}^{\infty} a_k^2 \{\sigma(k) - \sigma(k - 1 + 0)\} < \infty.$$

This is a generating function for the operator Q in the sense of our definition. Hence, the linear envelope of the set of functions $E(\Delta)\, g(t)$ coincides with the set of all step functions which are equal to zero outside of some interval, and this set is dense in $L_\sigma^2(-\infty, \infty)$ (cf. Section 42).

It is evident that a function $g(t) \in L_\sigma^2(-\infty, \infty)$ which is equal to zero on a set of positive σ-measure, cannot be a generator for the operator Q.

For a generating element one may take any function $g(t) \in L_\sigma^2(-\infty, \infty)$, which is distinct from zero everywhere with the possible exception of a set of zero σ-measure. For the proof of this assertion it is sufficient to establish that any function $f(t) \in L_\sigma^2(a, b)$ can be approximated with any accuracy by products of the form $g(t)\, q(t)$, where $q(t)$ is a step function:

$$\int_a^b | f(t) - g(t)\, q(t) |^2 d\sigma(t) < \epsilon^2. \tag{1}$$

We introduce the space L_ρ^2 (a, b), where

$$\rho(t) = \int_a^t | g(s) |^2\, d\sigma(s). \tag{2}$$

Since the function $\dfrac{f(t)}{g(t)}$ belongs to L_ρ^2 (a, b), there exists a step function $q(t)$ for which

$$\int_a^b \left| \frac{f(t)}{g(t)} - q(t) \right|^2 d\rho(t) < \epsilon^2.$$

After the replacement of $\rho(t)$ according to formula (2), we obtain (1).

THEOREM 1: *If the spectrum of a unitary operator U is simple and g is a generating element, then the linear envelope of the vectors $U^k g$ ($\pm\, k = 0, 1, 2, \ldots$) is dense in* H. *Conversely, if there exists a vector g such that the linear envelope of the vectors $U^k g$ ($\pm\, k = 0, 1, 2, \ldots$) is dense in* H, *then the spectrum of the operator U is simple and the vector g is a generating element.*

Proof: Let the spectrum of the operator U be simple and let g be a generating element. Assume that the linear envelope of the set of the vectors $U^k g$ ($\pm k = 0, 1, 2, \ldots$) is not dense in H. Then there exists a vector $h \neq 0$ for which

$$(U^k g, h) = 0 \qquad (\pm k = 0, 1, 2, \ldots).$$

From the integral representation of the unitary operator U it follows that

$$\int_0^{2\pi} e^{ikt} d(E_t g, h) = 0 \qquad (\pm k = 0, 1, 2, \ldots).$$

These equations and the uniqueness theorem yield

$$(E_t g, h) = 0 \qquad (0 \leqq t \leqq 2\pi).$$

This implies that the vector h is orthogonal to each vector of the form $E(\varDelta) g$. But this is impossible, since g is a generating element. Thus, the first part of the theorem is proved.

The second part is proved very easily in the same way.

THEOREM 2: (*The canonical form of a self-adjoint operator with a simple spectrum*). *Let A be a self-adjoint operator with a simple spectrum, g be a generating element, and $\sigma(t) = (E_t g, g)$. Then the formula*

$$f = \int_{-\infty}^{\infty} f(t)\, dE_t g$$

associates each function $f(t) \in L^2_\sigma(-\infty, \infty)$ with a vector $f \in$ H, and this correspondence is an isometric mapping of $L^2_\sigma(-\infty, \infty)$ into H. *It transforms the domain $\mathrm{D}_Q \subset L^2_\sigma(-\infty, \infty)$ of the multiplication operator Q in $L^2_\sigma(-\infty, \infty)$ into the domain $\mathrm{D}_A \subset$ H of the operator A. If the element $f \in \mathrm{D}_A$ corresponds to the function $f(t) \in L^2_\sigma(-\infty, \infty)$, then the element Af corresponds to the function $tf(t)$.*

Proof: We denote by G the set of all vectors of H which can be represented in the form (3). Since G is a linear manifold which contains all the vectors of the form

$$\int_{-\infty}^{\infty} \chi_\varDelta(t)\, dE_t g = E(\varDelta)\, g,$$

G is dense in H. If $f(t)$ belong to the set **K** which was introduced in Section 63, then

$$(E_tg, f) = \int_{-\infty}^{\infty} \overline{f(s)}\, d_s\,(E_tg, E_sg) = \int_{-\infty}^{t} \overline{f(s)}\, d(E_sg, g)$$

and

$$(f,f) = \int_{-\infty}^{\infty} f(t)\, d(E_tg,f) = \int_{-\infty}^{\infty} |f(t)|^2 d(E_tg, g) = \int_{-\infty}^{\infty} |f(t)|^2 d\sigma(t).$$

Thus, a linear manifold which is dense in $L^2_\sigma(-\infty, \infty)$ is transformed isometrically by (3) into a linear manifold which is dense in G. Therefore, the completeness of $L^2_\sigma(-\infty, \infty)$ implies that G is closed and, hence, G coincides with H, so that the first part of the theorem is proved.

In order to prove the second part of the theorem, we allow $f(t)$ to run through the set of all continuous functions which are equal to zero outside of a finite interval. If $f(t)$ is such a function and f is the vector in H which corresponds to it, then $f \in \mathrm{D}_A$ and by (3), for each $h \in \mathrm{H}$,

$$\begin{aligned}(Af, h) &= \int_{-\infty}^{\infty} t d(E_tf, h) = \int_{-\infty}^{\infty} t d(f, E_th) = \int_{-\infty}^{\infty} t d\left\{\int_{-\infty}^{\infty} f(s)\, d_s(E_sg, E_th)\right\} = \\ &= \int_{-\infty}^{\infty} t d\left\{\int_{-\infty}^{t} f(s)\, d(E_sg, h)\right\} = \int_{-\infty}^{\infty} t f(t)\, d(E_tg, h). \end{aligned} \tag{4}$$

It follows that

$$Af = \int_{-\infty}^{\infty} t\, f(t)\, dE_tg. \tag{5}$$

We let in (4)

$$h = Af,$$

and get

$$\| Af \|^2 = \int_{-\infty}^{\infty} | t\, f(t) |^2 d\sigma(t). \tag{6}$$

Recalling the definition of an operator represented by a Lebesgue-Stieltjes integral, we conclude from formulas (5) and (6) that an arbitrary function $f(t) \in L^2_\sigma(-\infty, \infty)$ belongs to the domain D_Q if and only if $f \in \mathrm{D}_A$. We see also that the application of the operator A to the vector f corresponds to the multiplication of the function $f(t)$ by t.

By the theorem just proved, each self-adjoint operator with a simple spectrum in a separable space is isomorphic to the operator of multiplication by the independent variable in $L^2_\sigma(-\infty, \infty)$. The latter will be referred to as the *canonical form* of the operator A.

This isomorphism determined by

$$f = \int_{-\infty}^{\infty} f(t)\, dE_t g \tag{3}$$

implies that

$$E(\Delta) f = \int_{\Delta} f(t)\, dE_t g \tag{3'}$$

first for continuous functions $f(t)$ which are equal to zero outside of a finite interval, and then for arbitrary $f(t) \in L^2_\sigma(-\infty, \infty)$. Choosing in (3) distinct generating elements g, we obtain an entire class of multiplication operators which are isomorphic to A. In the following section we characterize all distribution functions $\sigma(t) = (E_t g, g)$.

Here we note one general property of all generating elements g: every point of growth of the spectral function E_t is a point of growth of the function $(E_t g, g)$. The validity of this assertion follows from the fact that if Δ_0 is an interval of constancy of the function $(E_t g, g)$, then the linear manifold consisting of all the vectors $E(\Delta) g$, where Δ runs through the set of all real intervals, is orthogonal to the subspace $E(\Delta_0)$ H.

Let us recall the criterion given at the end of the previous section for a point λ to belong to the spectrum of a self-adjoint operator A which has a simple spectrum. In view of our present considerations, it is sufficient to verify this criterion for any one generating element.

We complete the present section with a proposition concerning self-adjoint operators which is the analogue of Theorem 1.

THEOREM 3: *If a self-adjoint operator A has a simple spectrum then there exists a vector h such that $A^k h$ is defined for $k = 0, 1, 2, \ldots$ and such that the linear envelope of this set of vectors is dense in* H. *Conversely, if there exists such a vector h, then the spectrum of the operator A is simple and the vector h is a generating element.*

Proof: The proof of the second part is very simple. Indeed, let the linear envelope of the set $A^k h$ $(k = 0, 1, 2, \ldots)$ be dense in H. The integral representation

$$(A^k h, f) = \int_{-\infty}^{\infty} t^k d(E_t h, f)$$

shows that there does not exist a vector $f \neq 0$ which is orthogonal to $E_t h$

for all t. Hence the linear envelope of the set $E(\Delta)h$ is dense in H, so that the spectrum of the operator is simple and h is a generating element.

Turning to the first part of the theorem, we prove that the vector h can be taken as

$$h = \int_{-\infty}^{\infty} e^{-t^2} dE_t g,$$

where g is some generating element of the operator. In this case,

$$A^k h = \int_{-\infty}^{\infty} t^k e^{-t^2} dE_t g \qquad (k = 0, 1, 2, \ldots).$$

If there exists a vector $f \neq 0$ which is orthogonal to all the vectors $A^k h$, then

$$\int_{-\infty}^{\infty} t^k e^{-t^2} f(t)\, d\sigma(t) = 0 \qquad (k = 0, 1, 2, \ldots),$$

where $\sigma(t) = (E_t g, g)$ and

$$0 \neq \int_{-\infty}^{\infty} |f(t)|^2 d\sigma(t) < \infty. \tag{7}$$

Letting

$$\omega(t) = \int_{-\infty}^{t} f(s)\, d\sigma(s) - C, \tag{8}$$

with a suitable choice of the constant C, we find that

$$\int_{-\infty}^{\infty} t^k e^{-t^2} \omega'(t)\, dt = 0 \qquad (k = 0, 1, 2, \ldots).$$

Since the system of Tchebysheff-Hermite functions is complete, it follows that $\omega(t) = 0$, which by (7) and (8) reduces to the contradiction

$$0 \neq \int_{-\infty}^{\infty} |f(t)|^2 d\,\sigma(t) = \int_{-\infty}^{\infty} \overline{f(t)}\, d\omega(t) = 0.$$

Orthogonalizing the sequence of vectors $\{A^k h\}_0^\infty$, we get an orthonormal basis $\{e_k\}_0^\infty$ all elements of which belong to D_A. Obviously, for $i > k + 1$,

$$(Ae_k, e_i) = 0,$$

and, since the operator A is symmetric, this equation holds also for $i < k - 1$.

Therefore, the matrix $((Ae_k, e_i))_{i,k=0}^{\infty}$ of the operator A relative to the basis $\{e_k\}_0^{\infty}$ is a Jacobi matrix. It is necessary to note that, in general, the constructed basis is not the basis of the matrix representative of A in the sense of Section 47.

70. Spectral Types

A *distribution function* is any left continuous nondecreasing function of bounded variation which is defined on the whole real axis. If $\sigma(t)$ is such a function, then $\sigma(\Delta) = \sigma(t'') - \sigma(t')$ (t' and t'' are the end points of Δ) is an *additive interval function.* (We also use the name distribution function for $\sigma(\Delta)$.)

Let us agree to say that a distribution function $\sigma(t)$ is *inferior*[10] to the distribution function $\rho(t)$ and write

$$\sigma(t) \prec \rho(t) \tag{1}$$

if $\sigma(t)$ is absolutely continuous with respect to $\rho(t)$, i.e., if for any $\Delta \subset [-\infty, \infty]$

$$\sigma(\Delta) = \int_{\Delta} \varphi(t)\, d\rho(t),$$

where $\varphi(t)$ is a ρ-measurable non-negative function.

If simultaneously with (1)

$$\rho(t) \prec \sigma(t) \tag{2}$$

holds, then the distribution functions $\sigma(t)$ and $\rho(t)$ are said to be of the same *spectral type.* If (1) holds but (2) does not, then we say that the spectral type of $\sigma(t)$ is less than the spectral type of $\rho(t)$.

Now let A be an arbitrary self-adjoint operator and E_t be its spectral function. The function $(E_t f, f)$, it is evident, is a distribution function in t for each fixed $f \in \mathrm{H}$. We call the spectral type of the function $(E_t f, f)$ the *spectral type of the element* f (with respect to the operator A). We say that the spectral types of the elements f belong to A. If among the elements $f \in \mathrm{H}$ there exists an element g of maximal spectral type with respect to A (i.e., an element g such that $(E_t f, f) \prec (E_t g, g)$ for all $f \in \mathrm{H}$), then we ascribe this spectral type to the operator A.

If A is a self-adjoint operator with a simple spectrum, then there exist elements of *maximal spectral type* with respect to A. This follows from the next theorem which, at the same time, gives the answer to the question of the preceding section about the characterization of the set of generating elements of an operator with a simple spectrum.

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[10] Translator's Note: Inferior has the meaning of "less than" in the partial ordering defined by the concept of absolute continuity.

THEOREM 1: *Let A be a self-adjoint operator with a simple spectrum. In order that an element g be a generating element it is necessary and sufficient that g is of maximal spectral type with respect to A.*

Proof: Let g be a generating element and f an arbitrary vector of H. In this case

$$f = \int_{-\infty}^{\infty} f(t)\, dE_t g,$$

where $f(t)$ is some function of $L^2_\sigma(-\infty, \infty)$ and $\sigma(t) = (E_t g, g)$. Furthermore, for each interval $\Delta \subset [-\infty, \infty]$,

$$E(\Delta) f = \int_\Delta f(t)\, dE_t g = \int_{-\infty}^{\infty} \chi_\Delta(t) f(t)\, dE_t g.$$

Thus, under the mapping of H into $L^2_\sigma(-\infty, \infty)$ determined by g, the vectors f and $E(\Delta) f$ correspond to the functions $f(t)$ and $\chi_\Delta(t) f(t)$. Since the mapping is an isometry,

$$(E(\Delta)f, f) = \int_{-\infty}^{\infty} |f(t)|^2 \chi_\Delta(t)\, d\sigma(t) = \int_\Delta |f(t)|^2 d\sigma(t).$$

Therefore,

$$(E_t f, f) \prec (E_t g, g),$$

so that the necessity is proved.

Let us assume now that g is an element of maximal spectral type, so that, for each $f \in$ H,

$$(E_t f, f) \prec (E_t g, g).$$

In particular, this relation holds for the generating element g_0, (which the operator A possesses since its spectrum is simple), i.e.,

$$(E_t g_0, g_0) \prec (E_t g, g).$$

On the other hand, by the first part of the theorem,

$$(E_t g, g) \prec (E_t g_0, g_0).$$

Thus, the elements g and g_0 are of the same spectral type, which implies that

$$\sigma(\Delta) \equiv (E(\Delta) g, g) = \int_\Delta p^2(t)\, d\sigma_0(t),$$

$$\sigma_0(\Delta) \equiv (E(\Delta) g_0, g_0) = \int_\Delta p_0^2(t)\, d\sigma(t),$$

where the functions $p(t) \geqq 0$ and $p_0(t) \geqq 0$ belong respectively to $L^2_{\sigma_0}(-\infty, \infty)$ and $L^2_{\sigma}(-\infty, \infty)$. These equations imply that the sets of zero σ-measure and zero σ_0-measure coincide. In addition, it is not difficult to see that

$$p(t)\, p_0(t) = 1$$

everywhere except a set of zero σ-measure. Therefore, each of the functions $p(t)$ and $p_0(t)$ can equal zero only on a set of zero σ-measure.

We must prove that g is a generating element for the operator A. In other words, we must show that the equation

$$(E(\Delta)\, g, h) = 0$$

is satisfied for every $\Delta \subset [-\infty, \infty]$ only when $h = 0$. But, by the isomorphism between H and $L^2_{\sigma_0}(-\infty, \infty)$,

$$(E(\Delta)\, g, h) = \int_{\Delta} g(t)\, \overline{h(t)}\, d\sigma_0(t),$$

where $g(t), h(t) \in L^2_{\sigma_0}(-\infty, \infty)$ and $|\, g(t)\,| = p(t)$. Hence, if g is not a generating element then, for some function $h(t) \in L^2_{\sigma_0}(-\infty, \infty)$ and for arbitrary $\Delta \subset [-\infty, \infty]$

$$\int_{\Delta} g(t)\, \overline{h(t)}\, d\sigma_0(t) = 0,$$

or, equivalently,

$$\int_{\Delta} p(t)\, \overline{h_0(t)}\, d\sigma_0(t) = 0.$$

This is impossible, since the linear envelope of the set of functions $\chi_{\Delta}(t)\, p(t)$, as we showed in the previous section, is dense in $L^2_{\sigma_0}(-\infty, \infty)$, since the function $p(t) \geqq 0$ can be zero only on a set of zero σ_0-measure.

In connection with the theorem just proved the question arises: is it possible to represent, in the form $(E_t f, f)$, every distribution function of a type which does not exceed the spectral type of A. An affirmative answer is given to this question in the following theorem.

THEOREM 2: *Let A be a self-adjoint operator with a simple spectrum and $\sigma(t)$ a given distribution function of a type which does not exceed the spectral type of A. Then there exists a vector $f \in$ H which this distribution function generates in the sense that*

$$\sigma(t) = (E_t f, f).$$

Proof: From the conditions of the theorem it follows that the distribution function $\sigma(t)$ can be represented in the form

$$\sigma(\Delta) = \int_{\Delta} \varphi(t)\, d(E_t g, g),$$

where g is a generating element and $\varphi(t) \geqq 0$. Let

$$f = \int_{-\infty}^{\infty} f(t)\, dE_t g = \int_{-\infty}^{\infty} \sqrt{\varphi(t)}\, dE_t g.$$

The element f generates the distribution function $\sigma(t)$, since

$$(E(\Delta)f, f) = \int_{\Delta} |f(t)|^2 d(E_t g, g) = \sigma(\Delta).$$

71. The Multiple Spectrum

If the linear envelope of the set of all eigenvectors of a self-adjoint operator is dense in the space, then it is natural to define the multiplicity (total multiplicity) of the spectrum of this operator as the maximal multiplicity of its eigenvalues.

Before we give the definition of the multiplicity of the spectrum of an arbitrary self-adjoint operator in H, we introduce the concept of the generating subspace. A subspace G is called a *generating subspace* of the self-adjoint operator A with the spectral function E_t if the linear envelope of the union of the sets $E(\Delta)$ G, where Δ runs through the set of all intervals, is dense in H.

DEFINITION 1: *The multiplicity (or total multiplicity) of the spectrum of a self-adjoint*[11] *operator A is the minimum dimension of the generating subspaces of this operator if A has a finite-dimensional generating subspace; otherwise, the multiplicity of the spectrum of A is infinite. The multiplicity of the operator A in the interval $\Delta_0 = [t', t'']$ is defined as the total multiplicity of the spectrum of the operator $E(\Delta_0) A$.*

It is easy to see that if G is a generating subspace of the operator $E(\Delta_0) A$, then for $\Delta_1 \subset \Delta_0$, the subspace $E(\Delta_1)$ G is a generating subspace of the operator $E(\Delta_1) A$. Therefore, for $\Delta_1 \subset \Delta_0$, the multiplicity of the operator A in the interval Δ_1 does not exceed its multiplicity in the interval Δ_0.

Hence, we are justified in making the next definition.

DEFINITION 2: *The multiplicity of the spectrum of a self-adjoint operator*

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[11] We omit the analogous definition for unitary operators. It is evident that the multiplicities of the spectra of a self-adjoint operator and of its Cayley transform coincide.

A at a point λ is the limit as $n \to \infty$ of the (monotone decreasing) sequence of the multiplicities of the spectrum of A in the intervals $\left[\lambda - \frac{1}{n}, \lambda + \frac{1}{n}\right]$.

It is not difficult to verify that, if λ is an isolated point of the spectrum, then Definition 2 coincides with the definition given earlier for the multiplicity of an eigenvalue. If the linear envelope of the set of all eigenvectors is dense in the space, then Definition 1 gives as the total multiplicity of the spectrum the maximal multiplicity of the eigenvalues.

It follows from Definition 1 that in a nonseparable space the total multiplicity of the spectrum of a self-adjoint operator is necessarily infinite.

In conclusion, we prove a lemma which will be used in the following section.

LEMMA: *Let a self-adjoint operator A have an n-fold spectrum and a generating subspace* G. *If the linear operator $A_0 \subset A$ is defined only on the linear envelope of the sets $E(\Delta)$* G, *where Δ runs through the set of all finite intervals, then the operator A is the closure of the operator A_0.*

Proof: Since the relation $A_0 \subset A$ is evident, we must show only that $A \subset \bar{A}_0$. With this aim we choose an arbitrary vector $f \in \mathrm{D}_A$ and let

$$f_k = E(\Delta_k)f \qquad (k = 1, 2, 3, \ldots),$$

where $\Delta_k = [-k, k]$. Since G is a generating subspace, the linear envelope of the sets $E(\Delta')$ G, where Δ' runs through all subintervals of the finite interval Δ, coincides with $E(\Delta)$ H. Therefore, the vectors f_k belong to the domain of A_0 and

$$A_0 f_k = AE(\Delta_k)f = E(\Delta_k)Af.$$

Let $k \to \infty$ to obtain

$$E(\Delta_k)f \to f, \; E(\Delta_k)Af \to Af.$$

Therefore, the vector f belongs to the domain of $\bar{A}_0$ and

$$A_0 f = Af,$$

so that the relation

$$A \subset \bar{A}_0$$

is proved.

72. The Canonical Form of a Self-Adjoint Operator with a Spectrum of Finite Multiplicity

In the present section we sketch a generalization of Theorem 2 of Section 69 for the case of an operator with a spectrum of finite multiplicity. With this aim we proceed to define the space $L^2_S(-\infty, \infty)$ of vector-valued functions, which is a generalization of the function space $L^2_\sigma(-\infty, \infty)$.

Let

$$S(t) = (\sigma_{ik}(t))_{i,k=1}^{m} \quad (m < \infty; \; -\infty < t < \infty)$$

be a Hermitean matrix function which satisfies the conditions:

1° *for arbitrary complex* ξ_k,

$$\sum_{i,k=1}^{m} \{\sigma_{ik}(t'') - \sigma_{ik}(t')\} \xi_i \bar{\xi}_k \geqq 0 \qquad (t' \leqq t''),$$

2° $$S(-\infty) = 0, \; S(t-0) = S(t).$$

A matrix function $S(t)$ *defined by conditions* 1° *and* 2° *we call a matrix distribution function and we write* $S(\Delta)$ *in place of* $S(t'') - S(t')$ *if* $\Delta = [t', t'']$.

From condition 1° it is easy to conclude that $\sigma_{ii}(t)$ $(i=1, 2, \ldots m)$ are nondecreasing functions of t and that the functions $\sigma_{ik}(t)$ $(i, k=1, 2, \ldots, m)$ are of bounded variation in every finite interval.

In the construction of the space $L_S^2(-\infty, \infty)$ we start with the set of all vector functions of the form

$$\overrightarrow{\chi(t)} = \{\chi_1(t), \chi_2(t), \ldots, \chi_m(t)\},$$

where $\chi_k(t)$ is the characteristic function of the interval $\Delta_k (k=1, 2, \ldots m)$. Such vector functions will be called *characteristic vector functions.* We form the linear envelope R of the set of all characteristic vector functions, letting

$$\alpha_1 \overrightarrow{\chi^{(1)}(t)} + \alpha_2 \overrightarrow{\chi^{(2)}(t)} = \{\alpha_1 \chi_1^{(1)}(t) + \alpha_2 \chi_1^{(2)}(t), \alpha_1 \chi_2^{(1)}(t) + \alpha_2 \chi_2^{(2)}(t), \ldots\}.$$

In terms of the matrix distribution function $S(t)$, we define a scalar product. First, let

$$(\overrightarrow{\chi^{(1)}(t)}, \overrightarrow{\chi^{(2)}(t)}) = \sigma_{ik}(\Delta_i^{(1)} \cap \Delta_k^{(2)}),$$

for vector functions of the form

$$\overrightarrow{\chi^{(1)}(t)} = \{0, \ldots, 0, \chi_i^{(1)}(t), 0, \ldots, 0\},$$

$$\overrightarrow{\chi^{(2)}(t)} = \{0, \ldots, 0, \chi_k^{(2)}(t), 0, \ldots, 0\},$$

where $\chi_i^{(1)}(t)$ and $\chi_k^{(2)}(t)$ are the characteristic functions of the intervals $\Delta_i^{(1)}$ and $\Delta_k^{(2)}$ and then, by linearity, extend the definition of the scalar product to the entire system R.[12]

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[12] Translator's Note: The closure of R in the norm introduced is L_S^2 $(-\infty, \infty)$.

The construction described does not give an analytic characterization of the space $L_S^2(-\infty, \infty)$. Moreover, this construction does not give us a justification for calling $L_S^2(-\infty, \infty)$ a space of vector functions, since those of its elements which do not belong to R are defined only as limits in norm and cannot be interpreted as vector functions on the basis of their definitions.

A functional analytic characterization of the elements of the space $L_S^2(-\infty, \infty)$ can be given,[13] in which the distribution function

$$v(t) = \sum_{i=1}^{m} \sigma_{ii}(t)$$

appears. There it is shown that the space $L_S^2(-\infty, \infty)$ consists of all vector functions

$$\overrightarrow{f(t)} = \{f_1(t), \ldots, f_m(t)\}$$

with components which are measurable and finite almost everywhere with respect to the measure induced by $v(t)$, and such that

$$\lim_{N\to\infty} (\overrightarrow{f_N(t)}, \overrightarrow{f_N(t)}) < \infty,$$

where

$$f_N(t) = \begin{cases} \overrightarrow{f(t)} & \text{if } \max\limits_{1 \leqq i \leqq m} |f_i(t)| < N, \\ 0 & \text{otherwise.} \end{cases}$$

However, for our purpose the functional analytic characterization of the space $L_S^2(-\infty, \infty)$ is not necessary. We remark only that our construction of $L_S^2(-\infty, \infty)$ reveals that this space contains the vector functions with continuous components which are zero outside of a finite interval. Therefore, we can define on the manifold of these vector functions

$$\overrightarrow{f(t)} = \{f_1(t), f_2(t), \ldots, f_m(t)\}$$

an operator Q_0 by the equation

$$Q_0\overrightarrow{f(t)} = \{tf_1(t), tf_2(t), \ldots, tf_m(t)\}.$$

The closure Q of the symmetric operator Q_0 we call the operator of multiplication by the variable t in the space $L_S^2(-\infty, \infty)$. With the aid of the remark at the end of Section 67 it is easy to establish that Q is a self-adjoint operator. Furthermore, Theorem 3 of Section 66 enables us to obtain the spectral function E_t of the operator Q. On the vector functions

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[13] cf., I. Kaz [1], Vol. I.

$$\overrightarrow{f(t)} = \{f_1(t), f_2(t), \ldots, f_m(t)\}$$

with continuous components which are equal to zero outside of a finite interval, the spectral function E_t is defined by the equation

$$E(\Delta)\overrightarrow{f(t)} = \{\chi_\Delta(t)f_1(t),\ \chi_\Delta(t)f_2(t),\ \ldots,\ \chi_\Delta(t)f_m(t)\},$$

where $\chi_\Delta(t)$ is the characteristic function of the interval Δ.

Repeating the argument of Section 69, we find that the multiplicity of the spectrum of the operator Q in $L_S^2(-\infty, \infty)$ does not exceed the order m of the matrix $S(t)$.

Now let A be a self-adjoint operator with a spectrum of multiplicity n ($n < \infty$) and the spectral function E_t. Let us agree to call every basis of any generating subspace of the operator A a *generating basis* of the operator. If $g_1, g_2, \ldots, g_m (m \geqq n)$ is a generating basis of the operator A then we denote the closure of the linear envelope of the set of all vectors $E(\Delta)g_i$, for fixed i, by G_i.

A basis of an n-dimensional generating subspace we call a *minimal generating basis.* It is possible to select a minimal generating basis so that the corresponding subspaces $\mathrm{G}_i (i = 1, 2, \ldots, n)$ are pairwise orthogonal. To obtain such a minimal basis $g_1, g_2, \ldots, g_n$ we start from an arbitrary basis $h_1, h_2, \ldots, h_n$ and, letting $g_1 = h_1$, we generate the subspace G_1. It is evident that h_2 is not in G_1 (since otherwise the multiplicity would be less than n) and, hence, $(E - P_{\mathrm{G}_1})h_2 \neq 0$. Letting $g_2 = (E - P_{\mathrm{G}_1})h_2$, we generate the subspace G_2. It is not difficult to see that $\mathrm{G}_2 \perp \mathrm{G}_1$. We continue this process in order to obtain the desired basis.

We do not presuppose that the generating basis $g_1, g_2, \ldots, g_m$ $(m \geqq n)$ is chosen such that the corresponding subspaces $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_m$ are pairwise orthogonal, and we restrict ourselves only to the requirement that these subspaces be linearly independent. In what follows, when speaking of a generating basis, we shall assume always that this last condition is satisfied.

It is easy to see that for any generating basis $g_1, g_2, \ldots, g_m$ $(n \leqq m < \infty)$, the matrix function

$$S(t) = (\,(E_t g_i, g_k)\,)_{i,k=1}^m$$

is a matrix distribution function. Thus, each generating basis defines some matrix distribution function.

Now it is possible to formulate the theorem about the canonical form of an operator with a spectrum of finite multiplicity.

THEOREM: *Let A be a self-adjoint operator in* H *with a multiple spectrum of multiplicity n. Let $g_1, g_2, \ldots, g_m$ $(n \leqq m < \infty)$ be an arbitrary generating basis and $S(t) = ((E_t g_i, g_k))_{i,k=1}^m$. Then there exists an isometric mapping of the space* H *into $L_S^2(-\infty, \infty)$ with the following properties. The domain $\mathrm{D}_A \subset \mathrm{H}$ of A is mapped into the domain $\mathrm{D}_Q \subset L_S^2$ $(-\infty, \infty)$ of the multiplication operator Q. If the element $f \in \mathrm{H}$ maps into the vector function $\overrightarrow{f(t)} \in L_S^2(-\infty, \infty)$, then the element Af maps into the vector function $Q\overrightarrow{f(t)}$.*

Proof: We associate with the vector

$$(1) \qquad f = E(\Delta_1)g_1 + E(\Delta_2)g_2 + \ldots + E(\Delta_m)g_m$$

of H the characteristic vector function

$$(2) \qquad \overrightarrow{f(t)} = \{\chi_{\Delta_1}(t), \chi_{\Delta_2}(t), \ldots, \chi_{\Delta_m}(t)\}$$

of $L_S^2(-\infty, \infty)$. Then

$$\|f\| = (f,f) = \left(\sum_{i=1}^m E(\Delta_i)g_i, \sum_{k=1}^m E(\Delta_k)g_k\right) =$$

$$= \sum_{i,k=1}^m (E(\Delta_i \cap \Delta_k)g_i, g_k) = \sum_{i,k=1}^m \sigma_{ik}(\Delta_i \cap \Delta_k) = \|\overrightarrow{f(t)}\|^2.$$

It is easy also to verify that to orthogonal vectors of the form (1) correspond orthogonal vector functions of the form (2). The extension of this correspondence by linearity and continuity to the whole space H is an isometric mapping of the space H into $L_S^2(-\infty, \infty)$. It is not difficult to verify also that for continuous vector functions $\overrightarrow{f(t)} = \{f_1(t), \ldots, f_m(t)\}$, with components equal to zero outside of a finite interval, formula (3) of Section 69 now takes the form

$$f = \sum_{i=1}^m \int_{-\infty}^{\infty} f_i(t)\, dE_t g_i.$$

We find next the vector function which corresponds to the vector Af, where f is a vector of the form (1). We have

$$Af = \int_{-\infty}^{\infty} t\, dE_t f = \int_{-\infty}^{\infty} t\, dE_t \left(\sum_{i=1}^m E(\Delta_i)g_i\right) = \sum_{i=1}^m \int_{\Delta_i} t\, dE_t g_i = \sum_{i=1}^m \int_{-\infty}^{\infty} t\chi_i(t)\, dE_t g_i.$$

If one takes into account the method of construction of the multiplication operator Q and the lemma of the preceding section, then the theorem on the canonical form is proved.

In Section 69 we showed that for every self-adjoint operator with a simple spectrum there exists an orthogonal basis with respect to which the matrix of the operator is a Jacobi matrix.

Completing the present section we show that this result generalizes to self-adjoint operators with spectra of finite multiplicity. Let A be such an operator. We select a generating basis $g_1, g_2, \dots g_n$ such that

$$\mathrm{H} = \mathrm{G}_1 \oplus \mathrm{G}_2 \oplus \mathrm{G}_3 \oplus \dots \oplus \mathrm{G}_n \quad (\mathrm{G}_i \perp \mathrm{G}_k,\ i \neq k).$$

Repeating the argument of Section 69, we get in each G_r a corresponding Jacobi matrix

$$(\,(Ae_{ri}, e_{rk})\,).$$

If we select in H a particular orthonormal basis $e_s (s=1, 2, 3, \dots)$, numbering the unit-vectors $e_{rk} (r=1, 2, 3, \dots n;\ k=1, 2, 3, \dots)$ first in the order of increasing first index and then in the order of increasing second index, i.e.,

$$e_1 = e_{11}, e_2 = e_{21}, \dots, n_n = e_{n1}\ ;\ e_{n+1} = e_{12}, e_{n+2} = e_{22}, \dots$$

then, with respect to the basis $\{e_s\}_{s=1}^{\infty}$ we have a generalized Jacobi matrix $(\,(Ae_i, e_k)\,)$. It will be of a special type since its only elements different from zero lie only on the principal diagonal and on the pair of nth neighboring diagonals.

73. Some Remarks about Unitary Invariants of Self-Adjoint Operators

Two operators A_1 *and* A_2 *which act respectively in the Hilbert spaces* H_1 *and* H_2 *are called*[14] *isomorphic* (*or unitarily equivalent*) *if there exists an isometric mapping* V *of the space* H_1 *into* H_2 *such that*

$$\text{(1)} \qquad \mathrm{D}_{A_2} = V\mathrm{D}_{A_1},$$

and

$$\text{(2)} \qquad A_2 = VA_1V^{-1}.$$

If the operator A_1 is symmetric, then the operator A_2 isomorphic to it is also symmetric. This follows immediately from (1) and (2).

The spectra of a pair of isometric operators coincide, since (1) and (2) yield

$$\text{(3)} \qquad \Delta_{A_2 - \lambda E_2} = (A_2 - \lambda E_2)\,\mathrm{D}_{A_2} = (VA_1V^{-1} - \lambda VV^{-1})\,\mathrm{D}_{A_2} = \\ = V(A_1 - \lambda E_1)\,\mathrm{D}_{A_1} = V\Delta_{A_1 - \lambda E_1}.$$

Moreover, it follows from the last equation that the total spectrum of A_1 and each of its parts (the point spectrum and the continuous spectrum) are *unitary invariants*, i.e., they do not change under the isomorphic transition from A_1 to A_2. Finally, it follows from the fact that the operator A_1 is self-adjoint that the operator A_2, isomorphic to A_1, is self-adjoint.

---

[14] We repeat here the definition which was given in Section 36.

We limit ourselves here to some remarks concerning unitarily equivalent self-adjoint operators. Let E_{1t} be the resolution of the identity of the operator A_1. Let

$$E_{2t} = VE_{1t}V^{-1}. \tag{4}$$

This formula defines a family of bounded self-adjoint operators in H_2 and it is easy to see that E_{2t} is a resolution of the identity in the space H_2. We verify, for example, that

$$E_{2u}E_{2v} = E_{2s},$$

where $s = \min\{u, v\}$. Hence,

$$E_{2u}E_{2v} = VE_{1u}V^{-1}VE_{1v}V^{-1} = VE_{1u}E_{1v}V^{-1} = VE_{1s}V^{-1} = E_{2s}.$$

We shall prove that E_{2t} is the decomposition of the identity of the operator A_2. With this aim, we consider the integral representation

$$(A_1f_1, g_1) = \int_{-\infty}^{\infty} t\,d(E_{1t}f_1, g_1).$$

Letting

$$f_1 = V^{-1}f_2,\quad g_1 = V^{-1}g_2,$$

we get

$$(VA_1V^{-1}f_2, g_2) = \int_{-\infty}^{\infty} t\,d(VE_{1t}V^{-1}f_2, g_2)$$

or

$$(A_2f_2, g_2) = \int_{-\infty}^{\infty} t\,d(E_{2t}f_2, g_2).$$

This is essentially what we set out to prove.

From relation (4) it follows that the multiplicity of the spectrum (the total multiplicity or multiplicity at a point) also is a unitary invariant of the self-adjoint operator.

If the linear envelope of the sequence of all eigenvectors of a self-adjoint operator A is dense in H, then the spectrum of A and the multiplicity of the spectrum at each point represent a complete system of unitary invariants of A, i.e., every other operator with identical spectrum and identical multiplicity of the spectrum at each point is isomorphic to A.

For self-adjoint operators with arbitrary spectra, the problem of finding a complete system of unitary invariants is extremely complicated, even in a separable space.

We show next how to solve this problem in the case of an arbitrary self-adjoint operator with a simple spectrum. Let A_1 and A_2 be two unitarily equivalent self-adjoint operators with simple spectra acting in the spaces H_1 and H_2, respectively. From the equation

$$(E_{2t}f_2,f_2) = (E_{1t}V^{-1}f_2,\ V^{-1}f_2) = (E_{1t}f_1,f_1),$$

it follows that the spectral types of the element f_1 with respect to A_1 and of the element $f_2 = Vf_1$ with respect to A_2 coincide. Therefore, the operators A_1 and A_2 are of the same spectral type.

The spectrum and multiplicity of the spectrum at each point do not represent complete systems of unitary invariants if the operator A has a continuous spectrum, as the following example indicates. The multiplication operator Q_{σ_1} in $L^2_{\sigma_1}(0, 1)$ and Q_{σ_2} in $L^2_{\sigma_2}(0, 1)$, where $\sigma_1(t) = t$ and $\sigma_2(t) = t^2$, have total spectra each of multiplicity one. But they are not isomorphic, since the spectral types of the distribution functions $\sigma_1(t)$ and $\sigma_2(t)$ do not coincide.

On the other hand, two operators with simple spectra and identical spectral types are isomorphic since, by Sections 69 and 70, they are both isomorphic to the same multiplication operator. Thus, the following theorem holds.

THEOREM: *In order that two operators with simple spectra be isomorphic it is necessary and sufficient that their spectral types coincide.*

For the extension from the case of simple spectra to the general case of multiple spectra, the determination of a complete system of unitary invariants is extremely complicated. The question of a complete system of unitary invariants of a self-adjoint operator with a multiple spectrum cannot be decided easily by a decomposition of such operators into orthogonal sums of operators with simple spectra, because such a decomposition is not defined uniquely. For the solution of the problem in the general case, one must choose the decomposition of the operator into an orthogonal sum in a special manner. This requires the introduction of the notion of the so-called *independent spectral types.*

The theory of unitary invariants of self-adjoint operators was developed by Hellinger for the case of a separable space and was developed later by A. I. Plesner for the case of a nonseparable space. The examination of these topics is beyond the scope of the present book. We recommend to the reader who desires to become acquainted with the theory of invariants the paper of A. I. Plesner and V. A. Rochlin in Vol. II of the journal *Uspekhi Math. Nauk* where this theory is presented with exhaustive completeness.

74. Some Remarks about Functions of Self-Adjoint Operators

In Section 63 we constructed the integral

$$(1) \qquad \int_{-\infty}^{\infty} \varphi(t)\, dE_t f = Tf$$

by starting from a given resolution of the identity $E_t(-\infty \leqq t \leqq \infty)$, an element $f \in \mathrm{H}$, and a function $\varphi(t)$ which belongs to $L^2_\sigma(-\infty, \infty)$, where $\sigma(t) = (E_t f, f)$. For a fixed function $\varphi(t)$, this integral defines an operator with domain D_T which consists of all vectors $f \in \mathrm{H}$ such that $\varphi(t)$ is measurable with respect to the distribution function $\sigma(t) = (E_t f, f)$ and

$$\int_{-\infty}^{\infty} |\varphi(t)|^2 d\sigma(t) < \infty.$$

We now consider the self-adjoint operator A to which the resolution of the identity E_t corresponds. If we take as the function $\varphi(t)$ some positive integer power of the variable t, then the operator T defined by formula (1) coincides[15] with the same power of the operator A. Therefore, if for $\varphi(t)$ we take the polynomial

$$p(t) = a_0 + a_1 t + \ldots + a_n t^n,$$

then the operator integral

$$\int_{-\infty}^{\infty} p(t)\, dE_t f$$

becomes

$$(a_0 E + a_1 A + \ldots + a_n A^n) f = p(A) f.$$

We see that a polynomial in a self-adjoint operator A, which has a direct definition, can also be defined as an operator integral with respect to the spectral function E_t of A. The latter method, i.e., the definition in terms of an integral, we can apply also in cases when an immediate definition of the function of an operator is not possible.

DEFINITION: *If $\varphi(t)$ is an arbitrary complex function defined on the whole real axis, then $\varphi(A)$ is the operator defined by the formula*

$$\varphi(A) f = \int_{-\infty}^{\infty} \varphi(t)\, dE_t f$$

---

[15] Cf. the end of Section 66.

for all vectors $f \in \mathrm{H}$ *for which* $\varphi(t)$ *is measurable with respect to the distribution function* $(E_t f, f)$ *and for which*

$$\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t f, f) < \infty.$$

We remark that the values of the function $\varphi(t)$ at regular points of the operator A do not affect the definition of the operator $\varphi(A)$. Therefore, we can assume that the function $\varphi(t)$ is defined only on the spectrum of the operator A.

For the construction of a productive theory of functions of operators it is expedient to introduce a somewhat restricted class of admissible functions $\varphi(t)$. We limit our considerations to functions $\varphi(t)$ which are measurable with respect to the distribution function $(E_t f, f)$ for any $f \in \mathrm{H}$. We denote this class of functions by $\mathbf{K}_0$.

Thus, let $\varphi(t) \in \mathbf{K}_0$. We prove first that the operator $\varphi(A)$ is linear. Evidently, it is sufficient to prove that if $f \in \mathrm{D}_{\varphi(A)}$ and $g \in \mathrm{D}_{\varphi(A)}$, then $f + g \in \mathrm{D}_{\varphi(A)}$. But, for arbitrary $h_1, h_2 \in \mathrm{H}$,

$$\|h_1 + h_2\|^2 + \|h_1 - h_2\|^2 = 2\|h_1\|^2 + 2\|h_2\|^2.$$

Therefore, for any $f, g \in \mathrm{H}$ and any $\Delta \subset [-\infty, \infty]$

$$(E(\Delta)\{f+g\}, f+g) \leqq 2(E(\Delta)f, f) + 2(E(\Delta)g, g),$$

and this implies that

$$\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t\{f+g\}, f+g) \leqq 2\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t f, f) + 2\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t g, g).$$

This proves our assertion.

THEOREM: *Let* $\varphi(t) \in \mathbf{K}_0$ *and*

$$\varphi(A) = \int_{-\infty}^{\infty} \varphi(t)\, dE_t.$$

Then, for each $f \in \mathrm{D}_{\varphi(A)}$ *and each* $h \in \mathrm{H}$,

$$(\varphi(A)f, h) = \int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, h).$$

Proof: First, we notice that for the functions $\varphi(t)$ in the class $\mathbf{K}$, introduced in Section 63, the theorem was proved in Section 63. Using this, it is easy to establish the theorem for any bounded function in the class $\mathbf{K}_0$. Indeed, let $\varphi(t)$ be such a function. We fix f and h, and let

$$\rho(t) = \tfrac{1}{4}(E_t\{f+h\}, f+h) + \tfrac{1}{4}(E_t\{f-h\}, f-h) +$$
$$+ \tfrac{1}{4}(E_t\{f+ih\}, f+ih) + \tfrac{1}{4}(E_t\{f-ih\}, f-ih) = (E_tf, f) + (E_th, h),$$

and introduce the space $L^2_\rho(-\infty, \infty)$ which is generated by the distribution function $\rho(t)$. Further, we construct a sequence $\{\psi_n(t)\}_1^\infty$ of functions in **K** which converge to $\varphi(t)$ in the metric of $L^2_\rho(-\infty, \infty)$. For each of the operators $\psi_n(A)$ the theorem is valid. Therefore,

$$(\psi_n(A)f, h) = \int_{-\infty}^{\infty} \psi_n(t)\, d(E_tf, h).$$

On the other hand,

$$\int_{-\infty}^{\infty} |\varphi(t) - \psi_n(t)|^2 d(E_tf, f) \leqq \int_{-\infty}^{\infty} |\varphi(t) - \psi_n(t)|^2 d\rho(t).$$

Hence, in view of the definition (cf. Section 63) of the integral

$$\int_{-\infty}^{\infty} \varphi(t)\, dE_tf$$

as the limit in the sense of strong convergence of

$$\int_{-\infty}^{\infty} \psi_n(t)\, dE_tf,$$

we conclude that

$$\lim_{n\to\infty} (\psi_n(A)f, h) = (\varphi(A)f, h).$$

It remains to prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \psi_n(t)\, d(E_tf, h) = \int_{-\infty}^{\infty} \varphi(t)\, d(E_tf, h). \tag{2}$$

But

$$\int_{-\infty}^{\infty} \{\varphi(t) - \psi_n(t)\}\, d(E_tf, h) =$$
$$= \tfrac{1}{4} \int_{-\infty}^{\infty} \{\varphi(t) - \psi_n(t)\}\, d\{\sigma_1(t) - \sigma_2(t) + i\sigma_3(t) - i\sigma_4(t)\},$$

where

$$\sigma_1(t) = (E_t\{f+h\}, f+h), \quad \sigma_2(t) = (E_t\{f-h\}, f-h),$$
$$\sigma_3(t) = (E_t\{f+ih\}, f+ih), \quad \sigma_4(t) = (E_t\{f-ih\}, f-ih).$$

Hence,

$$\left| \int_{-\infty}^{\infty} \{\varphi(t) - \psi_n(t)\} d(E_t f, h) \right| \leqq \int_{-\infty}^{\infty} | \varphi(t) - \psi_n(t) | d\rho(t) \leqq$$

$$\leqq \sqrt{\int_{-\infty}^{\infty} d\rho(t)} \sqrt{\int_{-\infty}^{\infty} | \varphi(t) - \psi_n(t) |^2 d\rho(t)},$$

and the limit relation (2) is proved. Thus, the theorem is proved for bounded functions in $\mathbf{K}_0$.

We assume now that $\varphi(t)$ is an unbounded function in $\mathbf{K}_0$. Let $\sigma(t) = (E_t f, f)$, where f is an arbitrary element in $D_{\varphi(A)}$ such that

$$\int_{-\infty}^{\infty} | \varphi(t) |^2 d\sigma(t) < \infty.$$

Letting

$$\varphi_n(t) = \begin{cases} \varphi(t) & \text{for} \quad |\varphi(t)| \leqq n, \\ 0 & \text{for} \quad |\varphi(t)| > n, \end{cases}$$

we obtain

$$(\varphi_n(A) f, h) = \int_{-\infty}^{\infty} \varphi_n(t)\, d(E_t f, h). \tag{3}$$

Now, since $\varphi_n(t)$ converges to $\varphi(t)$ in the metric of $L_\sigma^2(-\infty, \infty)$ as $n \to \infty$,

$$\varphi_n(A) f \to \varphi(A) f.$$

Hence, the right side of formula (3) has a limit as $n \to \infty$. According to the definition of the integral of an unbounded function, this limit is the integral

$$\int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, h).$$

Thus,

$$(\varphi(A) f, h) = \int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, h),$$

and the proof of the theorem is complete.

For bounded functions which belong to the class $\mathbf{K}_0$, it is easy to see that if

$$\varphi(t) = \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t),$$

then

$$\varphi(A) = \alpha_1\varphi_1(A) + \alpha_2\varphi_2(A),$$

where α_1 and α_2 are arbitrary numbers.

Furthermore, if $\varphi(t)$ is a bounded function in class $\mathbf{K}_0$, then $[\varphi(A)]^* = \bar{\varphi}(A)$, where $\bar{\varphi}$ denotes the function defined by $\bar{\varphi}(t) = \overline{\varphi(t)}$. Hence, for any $f, g \in \mathrm{H}$,

$$(4) \qquad (\varphi(A)f, g) = \int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, g) = \overline{\int_{-\infty}^{\infty} \bar{\varphi}(t)\, d(E_t g, f)} =$$

$$= \overline{(\bar{\varphi}(A)g, f)} = (f, \bar{\varphi}(A)g).$$

It is not difficult to verify that if $\varphi_1(t)$ and $\varphi_2(t)$ are bounded functions in class $\mathbf{K}_0$ and

$$\varphi(t) = \varphi_1(t)\, \varphi_2(t),$$

then

$$\varphi(A) = \varphi_1(A)\, \varphi_2(A).$$

In particular, it follows that each pair of bounded functions of a self-adjoint operator commute.

Another consequence of the facts established above is formulated: if $\varphi(t) \in \mathbf{K}_0$, then $\varphi(A)$ is a projection operator if and only if $\varphi(t)$ assumes only the two values zero and one, i.e., it is the characteristic function of some set $e \subset [-\infty, \infty]$ which is measurable with respect to the distribution function $(E_t f, f)$ for every $f \in \mathrm{H}$.

If $\chi_\Delta(t)$ is the characteristic function of some interval $\Delta \subset [-\infty, \infty]$, then

$$\chi_\Delta(A) = \int_{-\infty}^{\infty} \chi_\Delta(t)\, dE_t = \int_{\Delta} dE_t = E(\Delta).$$

Therefore, it seems natural to define $E(e)$, for an arbitrary set $e \subset [-\infty, \infty]$, which is measurable with respect to all distribution functions $(E_t f, f)$, by the formula

$$E(e) = \chi_e(t),$$

where $\chi_e(t)$ is the characteristic function of the set e.

We prove now that the relation

$$[\varphi(A)]^* = \bar{\varphi}(A).$$

holds not only for bounded functions in $\mathbf{K}_0$ but also for unbounded functions $\varphi(t) \in \mathbf{K}_0$ if the domain of the operator $\varphi(A)$ is dense in H (otherwise the operator $\varphi(A)$ would not have an adjoint). Let $g \in \mathrm{D}_{[\varphi(A)]^*}$, so that, for every $f \in \mathrm{D}_{\varphi(A)}$,

$$(\varphi(A)f, g) = (f, g^*). \tag{5}$$

Letting e_n denote the set of points on the real axis for which

$$|\varphi(t)| \leqq n,$$

we define

$$\varphi_n(t) = \begin{cases} \varphi(t) & (t \in e_n), \\ 0 & (t \notin e_n). \end{cases}$$

Since $\varphi_n(A)$ is bounded, we have, for every $h \in \mathrm{H}$,

$$(\varphi_n(A)h, g) = (h, \bar{\varphi}_n(A)g). \tag{6}$$

We define the vector f by

$$f = E(e_n)h.$$

It is not difficult to convince oneself that this vector belongs to $\mathrm{D}_{\varphi(A)}$. Indeed

$$E_t f = E_t E(e_n)h = \int_{-\infty}^{t} \chi_{e_n}(s)\, dE_s h,$$

and, therefore,

$$\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t f, f) = \int_{-\infty}^{\infty} |\varphi(t)|^2 \chi_{e_n}(t)\, d(E_t h, h) =$$

$$= \int_{-\infty}^{\infty} |\varphi_n(t)|^2 d(E_t h, h) < \infty.$$

By means of a similar calculation we find that, for each $h' \in \mathrm{H}$,

$$(\varphi(A)f, h') = \int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, h') = \int_{-\infty}^{\infty} \varphi(t)\, \chi_{e_n}(t)\, d(E_t h, h') =$$

$$= \int_{-\infty}^{\infty} \varphi_n(t)\, d(E_t h, h') = (\varphi_n(A)h, h'),$$

so that

$$\varphi(A)f = \varphi_n(A)h.$$

On the basis of (5) and (6), we conclude that

$$(h, \bar{\varphi}_n(A)g) = (E(e_n)h, g^*),$$

and, consequently,

$$(h, \bar{\varphi}_n(A)g) = (h, E(e_n)g^*).$$

Since h is an arbitrary vector,

$$\bar{\varphi}_n(A)\, g = E\,(e_n)\, g^*.$$

Therefore,

$$\| \bar{\varphi}_n(A)\, g \| = \| E\,(e_n)\, g^* \| \leqq \| g^* \|,$$

and

$$\int_{-\infty}^{\infty} | \bar{\varphi}_n\,(t) |^2 d\,(E_t g, g) \leqq \| g^* \|^2.$$

As n increases, the left side does not decrease. This implies that

$$\int_{-\infty}^{\infty} | \bar{\varphi}\,(t) |^2 d\,(E_t g, g) \leqq \| g^* \|^2.$$

In other words, it is proved that if $g \in D_{[\varphi(A)]^*}$ then $g \in D_{\bar{\varphi}(A)}$. But if $g \in D_{\bar{\varphi}(A)}$ then, for every $f \in D_{\varphi(A)}$,

$$(\varphi\,(A) f, g) = (f, \bar{\varphi}\,(A)\, g), \tag{7}$$

which may be proved in exactly the same way as equation (4). Comparing (5) and (7) we obtain the equation

$$g^* = \bar{\varphi}\,(A)\, g,$$

and our assertion is completely proved.

The product

$$\varphi_1(A)\, \varphi_2(A) \tag{8}$$

can be examined also in case the functions $\varphi_1(t)$, $\varphi_2(t) \in \mathbf{K}_0$ are not bounded. It is defined only for those elements $f \in D_{\varphi_2(A)}$ for which $\varphi_2(A) f \in D_{\varphi_1(A)}$. Therefore, generally speaking, it does not coincide with $\varphi\,(A)$, where $\varphi\,(t) = \varphi_1(t)\, \varphi_2(t)$. But, if an element f is in the domain of the operator (8), then, for every $h \in H$,

$$(\varphi_1(A)\, \varphi_2(A) f, h) = \int_{-\infty}^{\infty} \varphi_1(t)\, d(E_t \varphi_2(A) f, h)$$

and

$$(\varphi_2(A) f, E_t h) = \int_{-\infty}^{t} \varphi_2(s)\, d\,(E_s f, h),$$

so that

$$(\varphi_1(A)\, \varphi_2(A) f, h) = \int_{-\infty}^{\infty} \varphi_1(t)\, \varphi_2(t)\, d\,(E_t f, h) = \int_{-\infty}^{\infty} \varphi\,(t)\, d\,(E_t f, h) = (\varphi\,(A) f, h),$$

and, hence,

$$\varphi_1(A)\, \varphi_2(A) \subset \varphi\,(A).$$

We consider now the question of the inverse operators. The operator $\varphi(A)$, where $\varphi(t) \in \mathbf{K}_0$, has an inverse if and only if the relation

$$\varphi(A)f = 0,$$

or, equivalently,

$$\int_{-\infty}^{\infty} |\varphi(t)|^2 d(E_t f, f) = 0$$

is satisfied only for $f = 0$. This condition is satisfied if and only if the point set on which $\varphi(t) = 0$ has zero σ-measure for each distribution function $\sigma(t) = (E_t f, f), f \in \mathrm{H}$. Assume this condition and let

$$\psi(t) = \frac{1}{\varphi(t)}.$$

It is evident that the operator $\psi(A)$ is the inverse of the operator $\varphi(A)$. This justifies the equation

$$[\varphi(A)]^{-1} = \frac{1}{\varphi(A)}.$$

In light of the considerations of the present section, the Cayley transform of a self-adjoint operator A attains a new aspect. It is a linear fractional function of the operator A:

$$V = (A - iE)(A + iE)^{-1} = \frac{A - iE}{A + iE}.$$

The Cayley transform of a self-adjoint operator is a unitary operator. In general an operator $\varphi(A)$ will be unitary if and only if $|\varphi(t)| \equiv 1$. As an example, consider the operators

$$e^{isA} = \int_{-\infty}^{\infty} e^{ist} dE_t \qquad (-\infty < s < \infty),$$

which form the abelian group studied in Section 64.

Completing the present section, we consider what simplifications the assumption that the spectrum of the operator A is simple brings to the theory. Making this assumption, we let g denote some generating element of the operator and we let $\sigma(t)$ denote the distribution function $(E_t g, g)$. It is easy to see that the class $\mathbf{K}_0$ consists now of all functions which are measurable with respect to this particular distribution function. Indeed, let e be a set with finite σ-measure. This σ-measure equals

$$(9) \qquad \int_{-\infty}^{\infty} \chi_e(t)\, d\sigma(t),$$

where $\chi_e(t)$ is the characteristic function of the set e. If f is an arbitrary vector, then for any $\Delta \subset [-\infty, \infty]$

$$(10) \qquad (E(\Delta)f, f) = \int_{\Delta} p(t)\, d(E_t g, g),$$

where $p(t)$ is some non-negative σ-integrable function. Comparing (9) and (10), we see that the measure of the set e with respect to the distribution function $(E_t f, f)$ exists and equals

$$(E(e)f, f) = \int_{-\infty}^{\infty} \chi_e(t)\, d(E_t f, f) = \int_{-\infty}^{\infty} p(t)\, \chi_e(t)\, d(E_t g, g).$$

Let us take a particular function $\varphi(t) \in \mathbf{K}_0$ and construct the operator $\varphi(A)$. Each vector $f \in \mathrm{D}_{\varphi(A)}$ can be represented in the form

$$(11) \qquad f = \int_{-\infty}^{\infty} f(t)\, dE_t g,$$

where $f(t)$ belongs to $L^2_\sigma(-\infty, \infty)$. This representation shows that

$$f = f(A)g$$

so that

$$\varphi(A)f = \varphi(A)f(A)g = \int_{-\infty}^{\infty} \varphi(t) f(t)\, dE_t g.$$

The relation (11) maps H isomorphically into $L^2_\sigma(-\infty, \infty)$. Hence, the operator $\varphi(A)$ in H corresponds to the operator of multiplication by $\varphi(t)$ in $L^2_\sigma(-\infty, \infty)$.

75. Commutative Operators

In Section 14 we defined the commutativity of a pair of operators S and T, of which at least one (say, S) is bounded. According to this definition, S and T commute provided that if $f \in \mathrm{D}_T$ then $Sf \in \mathrm{D}_T$ and

$$STf = TSf.$$

Now we replace the operator T by an arbitrary self-adjoint operator A. We continue to let S be an arbitrary bounded operator. We shall prove that

S and A commute if and only if S and E_t commute for each real t, where E_t is the resolution of the identity for the operator A.

Thus, let the operators S and A commute. Then for each nonreal z and each $f \in \mathrm{D}_A$,

$$S(A - zE)f = (A - zE)Sf.$$

Consider the resolvent operator $R_z = (A - zE)^{-1}$ of A. Since the range of R_z is D_A, we have, for each $h \in \mathrm{H}$,

$$S(A - zE)R_z h = (A - zE)SR_z h$$

and

$$R_z Sh = SR_z h\,.$$

By the integral representation of the resolvent, we have, for each $g \in \mathrm{H}$,

$$\int_{-\infty}^{\infty} \frac{1}{t-z}\, d(E_t Sh, g) = \int_{-\infty}^{\infty} \frac{1}{t-z}\, d(SE_t h, g).$$

Since this relation holds for every nonreal z, it follows by means of an argument applied earlier that

$$E_t S = SE_t$$

for each $t \in [-\infty, \infty]$. Thus, one of the assertions is proved. The same formulas in the reverse order yield the other assertion.

From the considerations of the preceding section it follows that each bounded function of a self-adjoint operator of class $\mathbf{K}_0$ commutes with this operator. The converse proposition is not valid, i.e., the commutativity of a bounded operator S with a self-adjoint operator A, does not imply that S is a function of A. However, the following theorem holds.

THEOREM 1: *If a bounded operator S commutes with a self-adjoint operator A which has a simple spectrum, then S is a function of A.*

We shall prove this theorem below, but first we pause to consider a general question: what are necessary and sufficient conditions for a bounded operator S to be a function of a given self-adjoint operator A. One answer to this question is given in the following theorem.

THEOREM 2: (Riesz-Neumann) *A bounded operator S is a function of a self-adjoint operator A if and only if S commutes with every bounded operator T which commutes with A.*

The proof of the necessity of the criterion given in the theorem is very simple. Indeed, let

$$S = \varphi(A).$$

Let T be a bounded operator which commutes with A. Then

$$E_t T = TE_t$$

for each t. It follows that

$$(\varphi(A)Tf, g) = \int_{-\infty}^{\infty} \varphi(t)\, d(E_t Tf, g) = \int_{-\infty}^{\infty} \varphi(t)\, d(TE_t f, g) =$$

$$= \int_{-\infty}^{\infty} \varphi(t)\, d(E_t f, T^*g) = (\varphi(A)f, T^*g) = (T\varphi(A)f, g)$$

for all $f, g \in \mathrm{H}$. Therefore,

$$\varphi(A)T = T\varphi(A).$$

However, the proof of the sufficiency is not so simple. It will not be given in complete detail in this book. We restrict ourselves to the case for which the eigenvectors $e_k (k=1, 2, 3, \ldots)$ of the operator A form a complete orthonormal system in H (this condition is satisfied, for example, for a completely continuous operator in a separable space).

Let $P_k (k=1, 2, 3, \ldots)$ denote the operator of projection on the eigenvector e_k. Since the equations

$$Ae_k = \lambda_k e_k \qquad (k=1, 2, 3, \ldots)$$

imply that

$$AP_k = P_k A \qquad (k=1, 2, 3, \ldots),$$

we have, by the assumption of the theorem,

$$SP_k = P_k S.$$

The last equation implies that each eigenvector of A is an eigenvector of S, and that eigenvectors which belong to equal eigenvalues of A also belong to equal eigenvalues of S.

Let $\mu_k (k=1, 2, 3, \ldots)$ be the eigenvalues of S, so that

$$Se_k = \mu_k e_k \qquad (k = 1, 2, 3, \ldots).$$

The operators A and S are represented by the formulas

$$Af = \sum_{k=1}^{\infty} \lambda_k (f, e_k) e_k \qquad (f \in \mathrm{D}_A),$$

$$Sh = \sum_{k=1}^{\infty} \mu_k (h, e_k)\, e_k \qquad (h \in \mathrm{H}).$$

Since to equal values λ_k correspond equal values μ_k, we can define a function $\varphi(t)$ on the spectrum $\{\lambda_k\}_1^{\infty}$ by the equations

$$\varphi(\lambda_k) = \mu_k \qquad (k = 1, 2, 3, \ldots).$$

Thus, we obtain a representation of S in the form

$$Sh = \sum_{k=1}^{\infty} \varphi(\lambda_k)(h, e_k) e_k.$$

Thus,

$$S = \varphi(A).$$

Now we shall prove Theorem 1. Let g be a generating element of A. Then each vector $h \in \mathrm{H}$ is represented in the form

$$h = \int_{-\infty}^{\infty} h(t)\, d E_t g,$$

where $h(t)$ belongs to $L_\sigma^2(-\infty, \infty)$ with $\sigma(t) = (E_t g, g)$. In particular, we have the representation

$$Sg = \int_{-\infty}^{\infty} \varphi(t)\, dE_t g.$$

We prove that the function $\varphi(t)$, which appears here, is bounded. With this aim, we assume the contrary. Thus, we assume that there exists an infinite sequence of sets $e_1, e_2, e_3, \ldots$ of positive σ-measure such that

$$|\varphi(t)| > n$$

for $t \in e_n$. Consider the elements

$$f_n = E(e_n) g.$$

These elements are different from zero since

$$\|f_n\|^2 = (E(e_n) g, g) = m_\sigma\{e_n\} \neq 0.$$

Since the operator S commutes with A, it follows from the necessity of the criterion[16] of Theorem 2 that S commutes with every function of A and, in particular, with the operator

$$E(e_n) = \chi_{e_n}(A).$$

Thus

$$E(e_n) S = SE(e_n),$$

and, therefore,

$$\|Sf_n\|^2 = \|SE(e_n)g\|^2 = \|E(e_n)Sg\|^2 = (Sg, E(e_n)Sg) =$$

$$= \int_{-\infty}^{\infty} \varphi(t)\, d(E_t g, E(e_n)Sg) = \int_{-\infty}^{\infty} \chi_{e_n}(t)\, \varphi(t)\, d(E_t g, Sg) =$$

$$= \int_{-\infty}^{\infty} \chi_{e_n}(t)\, |\varphi(t)|^2 d(E_t g, g) > n^2 \int_{-\infty}^{\infty} \chi_{e_n}(t)\, d(E_t g, g) = n^2 \|f_n\|^2.$$

---

[16] Recall that the necessity was proved in complete detail.

This implies that the operator S is not bounded, which contradicts the original assumption. Therefore, the function $\varphi(t)$ is bounded. Now choose $f \in H$ arbitrarily and let

$$f = \int_{-\infty}^{\infty} f(t)\, d\, E_t g,$$

where $f(t) \in L_\sigma^2(-\infty, \infty)$. Then, for each $h \in H$,

$$(Sf, h) = (f, S^*h) = \int_{-\infty}^{\infty} f(t)\, d(E_t g, S^*h) =$$

$$= \int_{-\infty}^{\infty} f(t)\, d(SE_t g, h) = \int_{-\infty}^{\infty} f(t)\, d(E_t Sg, h) = \int_{-\infty}^{\infty} f(t)\, d(Sg, E_t h).$$

But since

$$(Sg, E_t h) = \int_{-\infty}^{\infty} \varphi(s)\, d_s(E_s g, E_t h) = \int_{-\infty}^{t} \varphi(s)\, d\,(E_s g, h),$$

we have

$$(Sf, h) = \int_{-\infty}^{\infty} f(t)\, \varphi(t)\, d\,(E_t g, h),$$

and, hence,

$$Sf = \int_{-\infty}^{\infty} \varphi(t) f(t)\, d\, E_t g.$$

By the concluding remarks of the preceding section, this implies that

$$S = \varphi(A),$$

so that Theorem 1 is proved.

Remark: Since each bounded operator T has the representation

$$T = B_1 + iB_2,$$

where B_1 and B_2 are bounded self-adjoint operators, it is sufficient in Theorem 2 to require that S commute with every bounded self-adjoint operator B which commutes with A.

76. Rings of Bounded Self-Adjoint Operators

A set $\mathfrak{M}$ of bounded self-adjoint operators $A, B, C, \ldots$, is called a *ring*, if $A, B \in \mathfrak{M}$ implies that $AB \in \mathfrak{M}$ and $\alpha A + \beta B \in \mathfrak{M}$ for arbitrary real α and β.

A ring is called *weakly closed* if $A_n \in \mathfrak{M}$ $(n=1, 2, 3, \ldots)$ and $A_n \overset{w}{\to} A$, imply that $A \in \mathfrak{M}$.

Since the product of two bounded self-adjoint operators is self-adjoint if and only if these operators commute, the operators in such a ring commute in pairs. Conversely, every set $\mathfrak{N}$ of pairwise commutative bounded self-adjoint operators determine a weakly closed ring, namely the *smallest weakly closed ring* which contains $\mathfrak{N}$. We donate this ring by $\mathfrak{R}(\mathfrak{N})$. It can be defined as the intersection of all weakly closed rings which contain $\mathfrak{N}$.

The following important theorem is due to von Neumann.

THEOREM 1: *For each weakly closed ring* $\mathfrak{M}$ *in a separable Hilbert space* H, *there exists a bounded self-adjoint operator* A *such that* $\mathfrak{M} = \mathfrak{R}(A)$.

Since the presentation of the proof of this theorem of von Neumann does not fall into the plan of the present book, we restrict ourselves to the statement of the theorem and to some of its implications.

Consider a finite or infinite set $\mathfrak{N}$ of pairwise commutative bounded self-adjoint operators in a separable space. We form the ring $\mathfrak{R}(\mathfrak{N})$. By the theorem of von Neumann, there exists a bounded self-adjoint operator A such that $\mathfrak{R}(\mathfrak{N}) = \mathfrak{R}(A)$. Let $\mathfrak{P}(A)$ denote the set of all bounded self-adjoint operators which commute with A. Let $\mathfrak{P}\{\mathfrak{P}(A)\}$ denote the set of all bounded self-adjoint operators which commute with each operator in $\mathfrak{P}(A)$. Since this set contains the operator A, every operator in $\mathfrak{P}\{\mathfrak{P}(A)\}$ commutes with A. Hence,

$$\mathfrak{P}\{\mathfrak{P}(A)\} \subset \mathfrak{P}(A).$$

We show next that $\mathfrak{P}\{\mathfrak{P}(A)\}$ is a weakly closed ring. Let $C', C'' \in \mathfrak{P}\{\mathfrak{P}(A)\}$. It is evident that the operator $\alpha' C' + \alpha'' C''$ belongs to $\mathfrak{P}\{\mathfrak{P}(A)\}$ for arbitrary real α' and α''. Since C' and C'' belong to $\mathfrak{P}\{\mathfrak{P}(A)\}$, they commute with each bounded self-adjoint operator which commutes with A. But, by Theorem 2 of the preceding section and the remark at the end of that section, C' and C'' are functions of the operator A, so that $C'C''$ is contained in $\mathfrak{P}\{\mathfrak{P}(A)\}$. Hence, $\mathfrak{P}\{\mathfrak{P}(A)\}$ is a ring.

It remains to show that the ring $\mathfrak{P}\{\mathfrak{P}(A)\}$ is weakly closed. Let

$$C_n \in \mathfrak{P}\{\mathfrak{P}(A)\}$$

for $n=1, 2, \ldots$ and, for arbitrary $f, g \in$ H, let

$$\lim_{n \to \infty} (C_n f, g) = (Cf, g).$$

If C' is an arbitrary operator in $\mathfrak{P}(A)$, then

$$(C_n C' f, g) = (C' C_n f, g) = (C_n f, C' g),$$

which yields

$$(CC'f, g) = (Cf, C'g).$$

It follows that

$$CC' = C'C,$$

so that the operator C commutes with every operator of $\mathfrak{P}(A)$. Therefore, $C \in \mathfrak{P}\{\mathfrak{P}(A)\}$ and $\mathfrak{P}\{\mathfrak{P}(A)\}$ is a weakly closed ring.

This result and the relation

$$A \in \mathfrak{P}\{\mathfrak{P}(A)\}$$

imply that

$$\mathfrak{R}(A) \subset \mathfrak{P}\{\mathfrak{P}(A)\}.$$

Since

$$\mathfrak{N} \subset \mathfrak{R}(\mathfrak{N}) = \mathfrak{R}(A),$$

we have

$$\mathfrak{N} \subset \mathfrak{P}\{\mathfrak{P}(A)\}.$$

Every operator in $\mathfrak{P}\{\mathfrak{P}(A)\}$ is, as we know, a function of the operator A. Hence, the same property is possessed by all the operators of $\mathfrak{N}$. Thus, we obtain the following theorem.

THEOREM 2: *If the bounded self-adjoint operators*

$$C', C'', C''', \ldots \tag{1}$$

in a separable space are pairwise commutative, then there exists a bounded self-adjoint operator A such that the operators (1) *are functions of A.*

There is a case for which this theorem can be proved simply without the application of Theorem 1, viz., if the number of operators (1) is finite and if the linear envelope of the eigenvectors of each of these operators is dense in H. We simplify the proof, without loss of generality, by assuming that the number of operators (1) is two. Thus, we have two commutative bounded self-adjoint operators L and M such that the linear envelope of the eigenvectors of each of these operators is dense in H.

Let E_s and F_t be the spectral functions for L and M, respectively. Further, let $\mathfrak{L}_\lambda$ and $\mathfrak{M}_\mu$ be the eigenmanifolds of these operators which correspond to the eigenvalues λ and μ, i.e.,

$$\mathfrak{L}_\lambda = (E_{\lambda+0} - E_\lambda)\,\mathrm{H},$$

$$\mathfrak{M}_\mu = (F_{\mu+0} - F_\mu)\,\mathrm{H}.$$

Since L and M commute,

$$(E_{\lambda+0} - E_\lambda)(F_{\mu+0} - F_\mu) = (F_{\mu+0} - F_\mu)(E_{\lambda+0} - E_\lambda).$$

Therefore,

$$P_{\lambda,\mu} = (E_{\lambda+0} - E_\lambda)(F_{\mu+0} - F_\mu)$$

is a projection operator for which

$$\mathrm{G}_{\lambda,\mu} = P_{\lambda,\mu}\,\mathrm{H} = \mathfrak{L}_\lambda \cap \mathfrak{M}_\mu.$$

It is easy to verify that if (λ', μ') and (λ'', μ'') are different pairs, then the subspaces $\mathrm{G}_{\lambda',\mu'}$, $\mathrm{G}_{\lambda'',\mu''}$ are orthogonal.

We prove that there does not exist a nonzero vector f which is orthogonal to all the subspaces $\mathrm{G}_{\lambda,\mu}$. Indeed, if $f \neq 0$, there exists at least one λ for which

$$(E_{\lambda+0} - E_\lambda)f \neq 0.$$

We denote this value λ by λ_0 and let

$$(E_{\lambda_0+0} - E_{\lambda_0})f = f'.$$

There exists μ_0 such that

$$(F_{\mu_0+0} - F_{\mu_0})f' \neq 0.$$

Hence,

$$P_{\lambda_0,\mu_0}f \neq 0,$$

which implies that the vector f is not orthogonal to the subspace $\mathrm{G}_{\lambda_0,\mu_0}$. Thus, our assertion is proved.

In each of the subspaces $\mathrm{G}_{\lambda,\mu}$ we construct a complete orthonormal system of vectors

$$e_{\lambda\mu}^{(1)},\ e_{\lambda\mu}^{(2)},\ e_{\lambda\mu}^{(3)},\ \ldots\ .$$

The set of all these vectors is countable, since the space H is separable. We enumerate this set of vectors $e_{\lambda\mu}^{(k)}$ and obtain orthonormal basis

$$e_1,\ e_2,\ e_3,\ \ldots$$

for H. Each vector e_k is an eigenvector of both of the operators L and M. Let

$$\left.\begin{aligned} Le_k &= \lambda_k e_k, \\ Me_k &= \mu_k e_k, \end{aligned}\right. \qquad (k = 1, 2, 3, \ldots).$$

Now we choose an arbitrary bounded sequence of real numbers $\{\alpha_k\}_1^\infty$ and define an operator A by the formula

$$Af = \sum_{k=1}^{\infty} \alpha_k(f, e_k)\, e_k.$$

Note that A is a bounded self-adjoint operator. Moreover, A is completely continuous if the sequence $\{\alpha_k\}_1^\infty$ converges to zero. The only restriction which we impose on the numbers α_k is that they all be distinct.

Let $\varphi(t)$ and $\psi(t)$ be real functions which satisfy the conditions

$$\varphi(\alpha_k) = \lambda_k, \quad \psi(\alpha_k) = \mu_k \qquad (k = 1, 2, 3, \ldots)$$

and otherwise are arbitrary. We find without difficulty that

$$L = \varphi(A), \qquad M = \psi(A).$$

77. Examples

We illustrate some of the general results of the present chapter in terms of the differential operator

$$P = i\frac{d}{dt}$$

and the operator Q of multiplication by an independent variable, both defined in $L^2(-\infty, \infty)$. These operators are self-adjoint. Furthermore, they are unitarily equivalent, i.e.,

$$P = \mathfrak{F}\,Q\,\mathfrak{F}^*,$$

where $\mathfrak{F}$ is the Fourier-Plancherel operator. Since the spectrum of the operator Q is simple and is the set of all real numbers (cf. Section 69), the spectrum of the operator P has the same properties.

A. Let $E_t^{(Q)}$ be the spectral function of the operator Q and $E_t^{(P)}$ that of the operator P. Then

$$E^{(Q)}(\Delta)\,h = \chi_\Delta(t)\,h(t),$$

where $\chi_\Delta(t)$ is the characteristic function of the interval Δ and $h = h(t)$ is an arbitrary element of the space $L^2(-\infty, \infty)$. Since (cf. Section 73)

$$E_t^{(P)} = \mathfrak{F}\,E_t^{(Q)}\mathfrak{F}^*,$$

we have, for each finite interval $\Delta = [\alpha, \beta]$,

$$E^{(P)}(\Delta)f = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\beta(u-t)} - e^{i\alpha(u-t)}}{i(u-t)}\,f(u)\,du,$$

where $f = f(t)$ is an arbitrary element of $L^2(-\infty, \infty)$.

B. We recall Theorem 2 of Section 69. By this theorem, if g is any generating element of the operator Q, then the formula

$$f = \int_{-\infty}^{\infty}\varphi(t)\,d\,E_t^{(Q)}g,$$

which associates the element $f = f(t) \in L^2(-\infty, \infty)$ with the element $\varphi(t) \in L_\sigma^2(-\infty, \infty)$, where $\sigma(t) = (E_t g, g)$, establishes an isometric mapping of $L^2(-\infty, \infty)$ into $L_\sigma^2(-\infty, \infty)$. Since Q is the multiplication operator in $L^2(-\infty, \infty)$, the question naturally arises as to the choice of a generating element g_0 for which this isometric mapping is the identity transformation:

$$f(s) = \int_{-\infty}^{\infty} f(t)\, d\, E_t^{(Q)} g_0$$

This equation holds if and only if

$$E^{(Q)}(\Delta)\, g_0 = \chi_\Delta(t),$$

where Δ is an arbitrary finite interval and $\chi_\Delta(t)$ is the characteristic function of Δ. But, as $\Delta \to [-\infty, \infty]$, the functions $\chi_\Delta(t)$ do not tend to a limit in $L^2(-\infty, \infty)$. Hence, the desired vector g_0 does not exist. Nevertheless, if we want an affirmative answer to our question, then we must augment the space $L^2(-\infty, \infty)$ by adding to it an *improper element* g_0 such that the projection of g_0 on each subspace $L^2(a, b)$, where $-\infty < a < b < \infty$, is the characteristic function of $[a, b]$. Thus, the element g_0 is a *unit function* which does not belong to the space $L^2(-\infty, \infty)$.

C. We consider now the operator P. Let $h = h\,(t)$ be an arbitrary generating element of this operator. Then, for each $f \in L^2(-\infty, \infty)$, there exists an element $\psi\,(t) \in L^2\,(-\infty, \infty)$, where $\sigma\,(t) = (E_t^{(P)}h, h)$, such that

$$f(s) = \int_{-\infty}^{\infty} \psi\,(t)\, d\, E_t^{(P)} h.$$

The question naturally arises as to the choice of a generating element h_0 for which $\psi\,(t)$ is the Fourier-Plancherel transform (or the inverse transform) of the function $f(t)$. We desire that

$$\psi\,(t) = \mathfrak{F}^* f(t).$$

The representations

$$f(s) = \int_{-\infty}^{\infty} \psi\,(t)\, d\, E_t^{(P)} h_0$$

and

$$f(s) = \text{l. i. m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi\,(t)\; e^{-ist} dt$$

yield

$$E^{(P)}\,(\Delta)\, h_0 = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\beta t} - e^{-i\alpha t}}{-it}$$

for each finite interval $\Delta = [\alpha, \beta]$. Thus, the desired vector h_0 should satisfy the condition

$$E^{(P)}(\Delta)\, h_0 = \mathfrak{F}\, E^{(Q)}(\Delta)\, g_0, \tag{1}$$

where g_0 is the identity function. Such a vector h_0 does not exist in $L^2(-\infty, \infty)$. We introduce h_0 as another improper element[17] of the space $L^2(-\infty, \infty)$. Equation (1) shows that the element h_0 should be considered as the Fourier-Plancherel transformation of the improper element g_0.

D. Turning to functions of the operator P, we should, first of all, define the class $\mathbf{K}_0$. This class is the set of all functions $\varphi(t)$ which are measurable on each finite real interval. We obtain the same class for the investigation of functions of the operator Q. Bounded functions are of greatest interest since they generate bounded operators.

Let $\varphi(t)$ be such a function. If $f = f(t)$ is an arbitrary function of $L^2(-\infty, \infty)$ and $g = g(t)$ is its inverse Fourier-Plancherel transform, then

$$\varphi(P) f = \mathfrak{F}\{\varphi(t)\, g(t)\}. \tag{2}$$

The situation is particularly simple when the function $\varphi(t)$ not only is bounded but also belongs to $L^2(-\infty, \infty)$. Indeed, then

$$\psi(t) = \mathfrak{F}\, \varphi(t)$$

exists and the right member of the equation is equal to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t-s) f(s)\, ds.$$

Thus, in this case,

$$\varphi(P) f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t-s) f(s)\, ds. \tag{3}$$

Since the right side of formula (2) becomes $\psi(t)$ when $g(t)$ is replaced by the identity function $g_0(t)$, $\psi(t)$ should be considered as the image of the improper element h_0 by the operator $\varphi(P)$:

$$\psi(t) = \varphi(P)\, h_0.$$

Thus, if $\varphi(t)$ is bounded and belongs to $L^2(-\infty, \infty)$ then it is sufficient to find $\varphi(P)\, h_0$, and the determination of $\varphi(P) f$ is reduced to finding some convolution according to (3).

Let us assume now that the function $\varphi(t)$ is bounded, but does not belong to $L^2(-\infty, \infty)$. Then it is natural to let

$$\varphi(t) = (t + i)\, \varphi_1(t),$$

---

[17] We remark that the introduction of improper elements is often useful not only in $L^2(-\infty, \infty)$ but also in other spaces.

where

$$\varphi_1(t) = \frac{\varphi(t)}{t+i}$$

and φ_1 belongs to $L^2(-\infty, \infty)$. Let

$$\varphi_1(P)\, h_0 = \psi_1(t).$$

Then

$$\varphi_1(P) f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_1(t-s) f(s)\, ds$$

and

$$\varphi(P) f = (P+i)\, \varphi_1(P) f = \frac{i}{\sqrt{2\pi}} \left(\frac{d}{dt} + 1\right) \int_{-\infty}^{\infty} \psi_1(t-s) f(s)\, ds.$$

E. Integral operators with kernels as functions of the difference of two arguments often occur in problems of analysis. We see that such operators are functions of the operator of differentiation. As an example, we introduce the operator defined by the formula

$$D_\lambda f = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \sqrt{\lambda}\,(t-s)}{t-s} f(s)\, ds,$$

where λ is a non-negative parameter. This operator plays an important role in the theory of the Fourier integral. It is of interest that D_λ is a resolution of the identity. This fact follows from the general circumstance: if E_t is the resolution of the identity for a self-adjoint operator A, then

$$E_{\sqrt{\lambda}} - E_{-\sqrt{\lambda}} \qquad (\lambda \geqq 0)$$

is the resolution of the identity for the operator A^2. But, it is easy to see that

$$\mathrm{D}_\lambda = E^{(P)}_{\sqrt{\lambda}} - E^{(P)}_{-\sqrt{\lambda}}.$$

Therefore, D_λ is the resolution of the identity for P^2. Now we show that the operator P^2 coincides with the operator L which is defined by the formula

$$Lf = -\frac{d^2 f}{dt^2}$$

for each function $f(t)$ in $L^2(-\infty, \infty)$ with a second derivative almost everywhere which also belongs to $L^2(-\infty, \infty)$.

To begin with, it is clear that

$$P^2 \subset L.$$

Therefore, it remains only to prove that if

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty, \quad \int_{-\infty}^{\infty} |f''(t)|^2\, dt < \infty,$$

then

$$\int_{-\infty}^{\infty} |f'(t)|^2\, dt < \infty.$$

It is sufficient to establish this fact for the case when the function $f(t)$ is real and the integration is over the positive real semi-axis. Consider the identity

$$\int_0^s [f'(t)]^2 dt = f(s)f'(s) - f(0)f'(0) - \int_0^s f(t)f''(t)\, dt,$$

$$\int_0^t f(s)f'(s)\, ds = \tfrac{1}{2}[f(t)]^2 - \tfrac{1}{2}[f(0)]^2.$$

Assume that $f'(t)$ does not belong to $L^2(-\infty, \infty)$. Then

$$\lim_{s\to\infty} \int_0^s [f'(t)]^2 dt = \infty.$$

From the first identity, of which the last term of the right member is bounded, we conclude that

$$\lim_{s\to\infty} f(s)f'(s) = \infty.$$

Therefore, from the second identity, we find that

$$\lim_{t\to\infty} [f(t)]^2 = \infty,$$

which contradicts the assumption that

$$\int_0^{\infty} [f(t)]^2 dt < \infty.$$

We proceed to prove that the multiplicity of the spectrum of the operator L equals two. First we show that this spectrum is not simple. Assuming the contrary we choose a generating element g of the operator L such that the linear envelope of the set of vectors $D_\lambda g (\lambda \geqq 0)$ is dense in $L^2(-\infty, \infty)$. Hence, the element g is *a fortiori* a generating element of P. But since the spectrum of P is the entire real line, the vector $E^{(P)}(\Delta)g$ can equal zero for no finite interval Δ. Let us take a finite interval $\Delta = [\alpha, \beta]$, where $\beta > \alpha > 0$, and represent the vector $E^{(P)}(\Delta)g$ in the form

$$E^{(P)}(\Delta)\, g = \int_0^\infty \varphi\,(t)\, d\mathrm{D}_t g,$$

which is possible since g is a generating element of L. This representation can be written in the form

$$\int_\alpha^\beta dE_t^{(P)} g = \int_0^\infty \varphi\,(s)\, d\,\{E^{(P)}_{\sqrt{s}} - E^{(P)}_{-\sqrt{s}}\}\, g = \int_{-\infty}^\infty \varphi\,(t^2)\, dE_t^{(P)} g$$

from which it follows that

$$\varphi\,(t^2) = \begin{cases} 1 & (\alpha < t < \beta), \\ 0 & (-\beta < t < -\alpha), \end{cases}$$

which is impossible.

Thus, the spectrum of the operator L is not simple. In order to prove that the multiplicity is two, it is necessary to show that the operator L has a generating basis which consists of two vectors. Let us choose two generating elements $h_1 = h_1(t)$ and $h_2 = h_2(t)$ of P, such that the first is an odd function and the second is an even function. We represent an arbitrary function $f(t)$ of $L^2(-\infty, \infty)$ in the form of the sum

$$f(t) = f_1(t) + f_2(t)$$

of an odd function $f_1(t)$ and an even function $f_2(t)$. Since h_1 is a generating element of P,

$$f_1 = \int_{-\infty}^\infty \varphi_1(s)\, d(E_s^{(P)} - E_0^{(P)}) h_1. \tag{4}$$

But

$$(E_s^{(P)} - E_0^{(P)})\, h_1 \equiv g_1(t; s) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{is(u-t)} - 1}{i\,(u - t)}\, h_1(u)\, du,$$

and, hence,

$$g_1(-t, -s) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{is(-u-t)} - 1}{i(u + t)}\, h_1(u)\, du =$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{is(u-t)} - 1}{i(-u + t)}\, h_1(-u)\, du =$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{is(u-t)} - 1}{i\,(u - t)}\, h_1(u)\, du = g_1(t; s).$$

Since $f_1(t)$ is odd, it follows from (4) that

$$f_1 = \int_{-\infty}^{\infty} \varphi_1(-s)\, d\,(E_s^{(P)} - E_0^{(P)})\, h_1.$$

This implies that $\varphi_1(t)$ is also an even function and the representation (4) takes the form

$$f_1 = \int_0^{\infty} \varphi_1(s)\, d\,(E_s^{(P)} - E_{-s}^{(P)})\, h_1 = \int_0^{\infty} \varphi_1(\sqrt{t})\, d\mathrm{D}_t h_1.$$

Analogously, it is proved that

$$f_2 = \int_0^{\infty} \varphi_2(\sqrt{t})\, d\mathrm{D}_t h_2.$$

Hence, the vectors h_1 and h_2 form a generating basis for the operator L. It may be observed that this basis is orthogonal.

Chapter VII

THEORY OF EXTENSIONS OF SYMMETRIC OPERATORS

78. Deficiency Indices

Let T be an arbitrary linear operator. For the present we require only that the domain of T be dense in H.

We call a complex number λ *a point of regular type of the operator* T *if there exists* $k = k(\lambda) > 0$ *such that for all* $f \in \mathrm{D}_T$

$$\|(T - \lambda E)f\| \geqq k\|f\|.$$

From this definition it follows that λ is a point of regular type of the operator T if and only if $(T - \lambda E)^{-1}$ exists and is bounded. In particular, the eigenvalues of the operator T are not points of regular type of T.

If λ_0 is a point of regular type and

$$|\lambda - \lambda_0| \leqq \delta \leqq \tfrac{1}{2}k(\lambda_0),$$

then, for each $f \in \mathrm{D}_T$,

$$\begin{aligned}\|(T - \lambda E)f\| &\geqq \|(T - \lambda_0 E)f\| - |\lambda - \lambda_0| \cdot \|f\| \geqq \\ &\geqq \{k(\lambda_0) - \delta\}\|f\| \geqq \tfrac{1}{2}k(\lambda_0)\|f\|.\end{aligned}$$

This fact, used once before (cf. the proof of Theorem 4, Section 43), shows that the set of points of regular type is open. We call this set of points the field of regularity of the operator T.

If A is a symmetric operator and $z = x + iy\ (y \neq 0)$, then (cf. the proof of Theorem 3, Section 43)

$$\|(A - zE)f\|^2 = \|(A - xE)f\|^2 + y^2\|f\|^2 \geqq y^2\|f\|^2$$

for each $f \in \mathrm{D}_A$. Hence, the upper and lower z-half-planes are connected subsets of the field of regularity of an arbitrary symmetric operator.

The field of regularity of an isometric operator V also contains two particular connected subsets: the region inside and the region outside of the unit circle. Indeed, for $|\zeta| < 1$

$$\|(V - \zeta E)f\| \geqq \|Vf\| - |\zeta| \cdot \|f\| = (1 - |\zeta|)\|f\|;$$

and, analogously, for $|\zeta| > 1$

$$\|(V - \zeta E)f\| \geqq |\zeta| \cdot \|f\| - \|Vf\| = (|\zeta| - 1)\|f\|.$$

THEOREM: *If Γ is a connected subset of the field of regularity of a linear operator T, then the dimension of the subspace* $\mathrm{H} \ominus \Delta_T(\lambda)$ *is the same for each* $\lambda \in \Gamma$.

Proof: We denote by P_λ the projection operator on the subspace

$$\mathfrak{N}_\lambda = \mathrm{H} \ominus \Delta_T(\lambda).$$

If we show that for arbitrary $\lambda_1, \lambda_2 \in \Gamma$,

$$\| P_{\lambda_1} - P_{\lambda_2} \| < 1, \tag{1}$$

then it follows from the theorem of Section 34 that the dimensions of the subspaces $\mathfrak{N}_{\lambda_1}$ and $\mathfrak{N}_{\lambda_2}$ are equal. In order to prove the inequality (1) for arbitrary $\lambda_1, \lambda_2 \in \Gamma$, it is sufficient, by using the Heine-Borel theorem, to show that for each $\lambda_0 \in \Gamma$ there exists $\delta = \delta(\lambda_0) > 0$ such that

$$\| P_\lambda - P_{\lambda_0} \| < 1$$

for $|\lambda - \lambda_0| < \delta$. Thus, let λ_0 be an arbitrary point of the region Γ and let

$$\delta = \delta(\lambda_0) \leqq \tfrac{1}{3} k(\lambda_0).$$

Since

$$k(\lambda_0) \|f\| \leqq \|(T - \lambda_0 E) f\| \leqq \|(T - \lambda E) f\| + |\lambda - \lambda_0| \cdot \|f\|,$$

we have, for $|\lambda - \lambda_0| \leqq \delta$,

$$\|(T - \lambda E) f\| \geqq \tfrac{2}{3} k(\lambda_0) \|f\|.$$

Let $|\lambda - \lambda_0| \leqq \delta$. Then, for each $h \in \mathfrak{N}_\lambda$ ($\|h\| = 1$),

$$\begin{aligned} \|(E - P_{\lambda_0}) h\| &= \sup_{f \in \mathrm{D}_T} \frac{|(h, \{T - \lambda_0 E\} f)|}{\|(T - \lambda_0 E) f\|} = \\ &= \sup_{f \in \mathrm{D}_T} \frac{|(h, \{T - \lambda E\} f + (\lambda - \lambda_0) f)|}{\|(T - \lambda_0 E) f\|} = \\ &= \sup_{f \in \mathrm{D}_T} \frac{|\lambda - \lambda_0| \cdot |(h, f)|}{\|(T - \lambda_0 E) f\|} \leqq \frac{1}{2} \end{aligned} \tag{2}$$

and, for each $h \in \mathfrak{N}_{\lambda_0}$ ($\|h\| = 1$),

$$\|(E - P_\lambda) h\| \leqq \tfrac{1}{2}. \tag{3}$$

By the second definition of the aperture of two linear manifolds (cf. Section 34) it follows from inequalities (2) and (3) that

$$\| P_\lambda - P_{\lambda_0} \| \leqq \tfrac{1}{2}.$$

Thus, the theorem is proved.

From our considerations, it follows that

$$\lim_{\lambda \to \lambda_0} \| P_\lambda - P_{\lambda_0} \| = 0 \qquad (\lambda, \lambda_0 \in \Gamma).$$

This relation expresses that as the point λ varies in the region Γ, the subspace $\mathfrak{N}_\lambda$ rotates continuously around the point $f = 0$.

We define the *deficiency* of a linear manifold as the dimension of its orthogonal complement $\mathfrak{N} = \mathrm{H} \ominus \mathfrak{M}$ and write

$$\operatorname{def} \mathfrak{M} = \dim \mathfrak{N}.$$

The deficiency can be finite or infinite.

The theorem just proved motivates the following definition.

DEFINITION: *The deficiency of the linear manifold* $\mathfrak{M}_\lambda = \Delta_T(\lambda)$ *for an arbitrary point* λ *of a given connected subset of the field of regularity of the operator* T *is called the deficiency number of the operator* T *with respect to this subset of the field of regularity. Furthermore,* $\mathfrak{N}_\lambda = \mathrm{H} \ominus \mathfrak{M}_\lambda$ *is called the deficiency subspace of the operator* T *for the point* λ *and each nonzero element of the deficiency subspace is called a deficiency element.*

Every symmetric operator A has two particularly important deficiency numbers, $\mathfrak{m}$ and $\mathfrak{n}$, corresponding to the lower and upper half-planes:

$$\operatorname{def} \Delta_A(z) = \begin{cases} \mathfrak{m} & (\Im z < 0), \\ \mathfrak{n} & (\Im z > 0). \end{cases}$$

Every isometric operator V has two analogous deficiency numbers, $\mathfrak{m}$ and $\mathfrak{n}$, corresponding to the regions inside and outside the unit circle:

$$\operatorname{def} \Delta_V(\zeta) = \begin{cases} \mathfrak{m} & (|\zeta| < 1), \\ \mathfrak{n} & (|\zeta| > 1). \end{cases}$$

These particular deficiency numbers are called the deficiency indices of the symmetric (respectively, isometric) operator. They are written in the form of an ordered pair $(\mathfrak{m}, \mathfrak{n})$.

From the theorem proved above immediately follow three propositions:

1° *If a symmetric operator has a real point of regular type, then its deficiency indices are equal:* $\mathfrak{m} = \mathfrak{n}$. *The same fact is true for an isometric operator if it has a point of regular type on the unit circle.*

2° *If* A *is a symmetric operator, then each nonreal number* z *is an eigenvalue of multiplicity* $\mathfrak{m}$ *for the self-adjoint operator* A^* *if* $\Im z > 0$, *and an eigenvalue of multiplicity* $\mathfrak{n}$ *if* $\Im z < 0$.

Indeed[1] if $f \in \mathrm{D}_A$ and $g \in \mathfrak{N}_{\bar{z}}$, then

$$(g, \{A - \bar{z}E\} f) = 0.$$

---

[1] Proposition 2° was first discovered by H. Weyl [2], Vol. I, for differential operators of the second order. Then T. Carleman proved it for integral operators. For arbitrary symmetric operators in H the proposition was established first by J. von Neumann. The proof of the more general theorem given earlier is due to M. G. Krein and M. A. Krasnoselski.

It follows that

$$(A^*g - zg, f) = 0,$$

so that

$$A^*g = zg.$$

Thus, z is an eigenvalue of the operator A^* with multiplicity $\dim \mathfrak{N}_{\bar{z}}$.

3° *The deficiency indices of an isometric operator V can be defined by means of the equations*

$$\mathfrak{m} = \operatorname{def} \mathrm{D}_V,$$

$$\mathfrak{n} = \operatorname{def} \Delta_V.$$

It is enough to prove just the first equation. For each $\zeta \neq 0$,

$$\Delta_V(\zeta) = (V - \zeta E)\mathrm{D}_V = \left(\frac{1}{\zeta}V - E\right)\mathrm{D}_V =$$

$$= \left(\frac{1}{\zeta}E - V^{-1}\right)V\mathrm{D}_V = \left(V^{-1} - \frac{1}{\zeta}E\right)\mathrm{D}_{V^{-1}} = \Delta_{V^{-1}}\left(\frac{1}{\zeta}\right).$$

If $|\zeta| > 1$, then

$$\mathfrak{m} = \operatorname{def} \Delta_V(\zeta) = \operatorname{def} \Delta_{V^{-1}}\left(\frac{1}{\zeta}\right) =$$

$$= \operatorname{def} \Delta_{V^{-1}}(0) = \operatorname{def} \Delta_{V^{-1}} = \operatorname{def} \mathrm{D}_V.$$

79. Further Remarks on the Cayley Transform

In Section 67 we introduced the idea of the Cayley transform V of a closed symmetric operator A with the aid of the formulas

$$(1) \qquad \begin{cases} (A - \bar{z}E)h = f, \\ (A - zE)h = Vf, \end{cases}$$

where z is any nonreal number and $h \in \mathrm{D}_A$. We assume in the present section that $\Im z > 0$. The operator V is expressed by the operator A in the form

$$Vf = (A - zE)(A - \bar{z}E)^{-1}f,$$

where the domain D_V of the operator V is $\Delta_A(\bar{z})$. By formula (1),

$$(2) \qquad \begin{cases} h = \dfrac{(V - E)f}{\bar{z} - z}, \\[2ex] Ah = \dfrac{(\bar{z}V - zE)f}{\bar{z} - z}, \end{cases}$$

and, therefore,

$$Ah = (\bar{z}V - zE)(V - E)^{-1}h.$$

For what follows, it is very important that the deficiency indices $(\mathfrak{m}, \mathfrak{n})$ of the operator A are the same as the deficiency indices of the operator V. According to the definition,

$$\mathfrak{m} = \operatorname{def} \Delta_A(\bar{z}).$$

But

$$\Delta_A(\bar{z}) = \mathrm{D}_V,$$

so that

$$\operatorname{def} \mathrm{D}_V = \mathfrak{m}.$$

On the other hand,

$$\mathfrak{n} = \operatorname{def} \Delta_A(z)$$

by definition, and

$$\Delta_A(z) = \Delta_V,$$

so that

$$\operatorname{def} \Delta_V = \mathfrak{n}.$$

It remains to apply proposition 3 of the preceding section.

THEOREM 1: *If V is an isometric operator and if the manifold $\Delta_V(1)$ is dense in* H, *then the operator A which is defined by formula* (2′) *is symmetric and the operator V is its Cayley transform.*

Proof: Since $\Delta_V(1)$ is dense in H, the inverse operator $(V - E)^{-1}$ exists. Indeed, if this operator does not exist, then unity is an eigenvalue of the operator V. But if

$$Vg = g \qquad (g \neq 0)$$

then, for each $f \in \mathrm{D}_V$,

$$(Vf - f, g) = (Vf, g) - (f, g) = (Vf, Vg) - (f, g) = 0,$$

i.e., $g \perp \Delta_V(1)$.

Since the operator $(V - E)^{-1}$ exists, the operator

$$A = (\bar{z}V - zE)(V - E)^{-1}$$

exists and its domain is dense in H. We shall show that this operator is symmetric.

Let f and g be arbitrary elements of $\mathrm{D}_A = \Delta_V(1)$:

$$\left.\begin{aligned} f &= V\varphi - \varphi, \\ g &= V\psi - \psi, \end{aligned}\right. \qquad (\varphi, \psi \in \mathrm{D}_V).$$

Then

$$Af = (\bar{z}V - zE)\varphi = \bar{z}V\varphi - z\varphi,$$
$$Ag = (\bar{z}V - zE)\psi = \bar{z}V\psi - z\psi.$$

Therefore,

$$\begin{aligned}(Af, g) &= (\bar{z}V\varphi - z\varphi, V\psi - \psi) = \\ &= (z + \bar{z})(\varphi, \psi) - \bar{z}(V\varphi, \psi) - z(\varphi, V\psi)\end{aligned}$$

and

$$\begin{aligned}(f, Ag) &= (V\varphi - \varphi, \bar{z}V\psi - z\psi) = \\ &= (z + \bar{z})(\varphi, \psi) - \bar{z}(V\varphi, \psi) - z(\varphi, V\psi),\end{aligned}$$

so that

$$(Af, g) = (f, Ag).$$

The proof of the relation

$$V = \frac{A - zE}{A - \bar{z}E}$$

is not difficult. Thus, the operator V is the Cayley transform of the operator A.

In what follows we apply the name "Cayley transform" to each of the two operators V and A which are connected by the relations (1) and (2), i.e., we not only call the operator V the Cayley transform of the operator A but also we call the operator A the Cayley transform of the operator V.

From the propositions proved above immediately follows the next theorem.

THEOREM 2: *Let A_1 and A_2 be symmetric operators and let V_1 and V_2 be their Cayley transforms. In order that A_2 be an extension of A_1 it is necessary and sufficient that V_2 be an extension of V_1.*

By Theorem 2 the question of symmetric extensions of a given operator A is reduced to the question of isometric extensions of its Cayley transform V.

From Section 9 we know that closed linear manifolds F and G can be the domain and range, respectively, of an isometric operator if and only if their dimensions are equal. In the following manner the isometric extensions of an operator V can be obtained.

In the deficiency subspaces $\mathrm{H} \ominus \mathrm{D}_V$ and $\mathrm{H} \ominus \Delta_V$ we choose two subspaces F and G of the same dimensions and we construct an arbitrary isometric operator V' with domain F and range G. We further define a linear operator $\tilde{V}$ with domain $\mathrm{D}_{\tilde{V}} = \mathrm{D}_V \oplus \mathrm{F}$ and range $\Delta_{\tilde{V}} = \mathrm{D}_V \oplus \mathrm{G}$ by the formulas

$$\tilde{V}f = \begin{cases} [Vf,\ f \in \mathrm{D}_V], \\ [V'f, f \in \mathrm{F}]. \end{cases}$$

It is evident that $\tilde{V}$ is an isometric extension of V, and for all possible choices of F, G, and V', we get all isometric extensions $\tilde{V}$ of the operator V and each one just once.

In order to find a symmetric extension $\tilde{A}$ of the operator A we pass to the Cayley transform of the operator, find an extension $\tilde{V}$ of the operator V by the method described above and, finally, let $\tilde{A}$ be the Cayley transform of $\tilde{V}$. The formula which corresponds to this procedure will be given in the following section.

From the argument outlined above it follows, in particular, that the operator A is a maximal symmetric (self-adjoint) operator if and only if its Cayley transform V is a maximal isometric (correspondingly, unitary) operator. Therefore, we have

THEOREM 3: *In order that a symmetric operator be maximal, it is necessary and sufficient that one of its deficiency indices equal zero. In order that a symmetric operator be self-adjoint, it is necessary and sufficient that both its deficiency indices equal zero.*

By the procedure described above, we also obtain

THEOREM 4: *Let A be an arbitrary symmetric operator with deficiency indices* $(\mathfrak{m}, \mathfrak{n})$. *Then A has a maximal symmetric extension. If* $\mathfrak{m} \neq \mathfrak{n}$, *then none of the extensions is self-adjoint. If* $\mathfrak{m} = \mathfrak{n} < \infty$, *then every maximal extension of the operator A is self-adjoint. If the deficiency indices* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are infinite and equal, then A has maximal self-adjoint extensions and non-self-adjoint extensions.*

80. The Neumann Formulas

We begin with some remarks about linear independence. In the very beginning of the book we introduced the concept of a linearly independent set of vectors. Now we introduce two more general concepts.

Let us agree to call the linear manifolds $\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n$ $(n < \infty)$ linearly independent if the equation

$$f_1 + f_2 + \ldots + f_n = 0 \qquad (f_k \in \mathrm{M}_k;\ k = 1, 2, \ldots, n)$$

implies that

$$f_1 = f_2 = \ldots = f_n = 0.$$

If the manifolds $M_1, M_2, \ldots, M_n$ are linearly independent, then it is possible to form their direct sum

$$M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

(we defined this in Section 7).

Further, let M and $\tilde{M}$ be two linear manifolds such that $M \subset \tilde{M}$. The vectors $f_1, f_2, \ldots, f_k$ of $\tilde{M}$ are called *linearly independent modulo* M if from

$$a_1 f_1 + a_2 f_2 + \ldots + a_k f_k \in M,$$

it follows that

$$a_1 = a_2 = \ldots = a_k = 0.$$

It is evident that a set of vectors in $\tilde{M}$ which are linearly independent modulo M is also linearly independent in the ordinary sense.

The *dimension of* $\tilde{M}$ *modulo* M we define as the maximal number m of vectors in M which are linearly independent modulo M. We write

$$\dim \tilde{M} = m \qquad (\mathrm{mod}\ M).$$

If there are arbitrary many vectors in $\tilde{M}$ which are linearly independent vectors modulo M, then we write

$$\dim \tilde{M} = \infty \qquad (\mathrm{mod}\ M).$$

It is evident that the dimension of $\tilde{M}$ modulo M does not exceed the ordinary dimension of $\tilde{M}$.

We express the relation $f \in M$ also by

$$f \equiv 0 \qquad (\mathrm{mod}\ M)\ ;$$

and then the equation

$$f \equiv g \qquad (\mathrm{mod}\ M),$$

means that $f - g \in M$.

The dimension of $\tilde{M}$ modulo M defined above is the ordinary dimension of the quotient manifold $\tilde{M}/M$.

THEOREM: *Let A be an arbitrary symmetric operator with domain* D_A *and let* $\mathfrak{N}_{\bar{z}}$ *and* $\mathfrak{N}_z$ $(\Im z > 0)$ *be any pair of its deficiency subspaces. Then the domain* D_{A^*} *of the operator* A^* *has the following representation as the direct sum of three linear manifolds:*

$$D_{A^*} = D_A \oplus \mathfrak{N}_{\bar{z}} \oplus \mathfrak{N}_z.$$

Proof: Since the manifold D_{A^*} is linear and contains D_A, $\mathfrak{N}_{\bar{z}}$, and $\mathfrak{N}_z$,

$$D_{A^*} \supset D_A \oplus \mathfrak{N}_{\bar{z}} \oplus \mathfrak{N}_z.$$

We show that, conversely, each $f \in D_{A^*}$ can be represented in the form

$$(1) \qquad f = f_0 + g_z + g_{\bar z},$$

where[2] $f_0 \in D_A$, $g_z \in \mathfrak{N}_{\bar z}$ and $g_{\bar z} \in \mathfrak{N}_z$. Note that (1) yields

$$(1') \qquad A^*f = Af_0 + zg_z + \bar z g_{\bar z}.$$

Let $f \in D_{A^*}$. We decompose $A^*f - zf$ into its components in the orthogonal subspaces $\mathfrak{M}_z$ and $\mathfrak{N}_z$:

$$A^*f - zf = (Af_0 - zf_0) + (\bar z - z)\, g_{\bar z}.$$

But $A^*g_{\bar z} = \bar z g_{\bar z}$; therefore

$$A^*(f - f_0 - g_{\bar z}) = z(f - f_0 - g_{\bar z}).$$

We conclude that

$$f - f_0 - g_{\bar z} \in \mathfrak{N}_{\bar z},$$

i.e., that

$$f - f_0 - g_{\bar z} = g_z,$$

or

$$f = f_0 + g_{\bar z} + g_z.$$

To complete the proof of the theorem it remains to show that the representation (1) of each element $f \in D_{A^*}$ is unique. Assuming the contrary, let us suppose that

$$(2) \qquad f_0 + g_z + g_{\bar z} = 0.$$

Applying the operator A^* to both sides of this equation, we get

$$(2') \qquad Af_0 + zg_z + \bar z g_{\bar z} = 0.$$

Multiply (2) by z and subtract from (2′) to obtain

$$Af_0 - zf_0 + (\bar z - z)\, g_{\bar z} = 0.$$

It follows from the orthogonality of the summands that $(\bar z - z)\, g_{\bar z} = 0$. In the same way we get $(\bar z - z)\, g_z = 0$. Hence

$$f_0 = g_{\bar z} = g_z = 0,$$

and the theorem is proved.

From formulas (1) and (1′) it follows that, for each $f \in D_{A^*}$,

$$\begin{aligned}(A^*f, f) &= (Af_0 + zg_z + \bar z g_{\bar z}, f_0 + g_z + g_{\bar z}) = \\ &= (Af_0, f_0) + [(Af_0, g_z) + z(g_z, f_0)] + [(Af_0, g_{\bar z}) + \bar z(g_{\bar z}, f_0)] + \\ &\qquad + [z(g_z, g_{\bar z}) + \bar z(g_z, g_z)] + [z(g_z, g_z) + \bar z(g_{\bar z}, g_z)] = \\ &= (Af_0, f_0) + [\bar z(f_0, g_z) + z(g_z, f_0)] + [z(f_0, g_{\bar z}) + \bar z(g_{\bar z}, f_0)] + \\ &\qquad + [\bar z(g_{\bar z}, g_z) + z(g_z, g_{\bar z})] + [z\|g_z\|^2 + \bar z\|g_{\bar z}\|^2].\end{aligned}$$

---

[2] We recall (cf. item 2°, Section 78) that $\mathfrak{N}_z$ is the eigenmanifold of the operator A^* associated with the eigenvalue z. Therefore, elements from $\mathfrak{N}_{\bar z}$ we denote by g_z.

Since the first four summands are real,

$$(3) \qquad \Im\,(A^*f,f) = \Im z\,(\,\|g_z\|^2 - \|g_{\bar z}\|^2).$$

In correspondence with formula (3), the domain D_{A^*} consists of three (nonlinear) manifolds: Γ^+ (the set of all elements f for which $\Im\,(A^*f,f) > 0$), Γ^- (the set of elements f for which $\Im\,(A^*f,f) < 0$) and Γ^0 (the set of elements f, for which (A^*f,f) is real). The element

$$f = f_0 + g_z + g_{\bar z}$$

belongs to Γ^+, Γ^-, or Γ^0 according as

$$\|g_z\| > \|g_{\bar z}\|, \quad \|g_z\| < \|g_{\bar z}\|, \quad \text{or} \quad \|g_z\| = \|g_{\bar z}\|.$$

We derive next a representation similar to (1) for the domain $\mathrm{D}_{\tilde A}$ of an arbitrary symmetric extension $\tilde A$ of the operator A.

In order to indicate the dependence on z of the subspaces F and G, which were introduced in Section 79, we write F_z and G_z. Thus $\mathrm{F}_z \subset \mathfrak{N}_{\bar z}$ and $\mathrm{G}_z \subset \mathfrak{N}_z$.

From the considerations of Section 79 it follows that

$$\begin{aligned}\mathrm{D}_{\tilde A} &= (\tilde V - E)\,\mathrm{D}_{\tilde V} = (\tilde V - E)(\mathrm{D}_V \oplus \mathrm{F}_z) = \\ &= (V - E)\,\mathrm{D}_V \oplus (V' - E)\,\mathrm{F}_z = \mathrm{D}_A \oplus (V' - E)\,\mathrm{F}_z\end{aligned}$$

and

$$\mathrm{D}_{\tilde A} = \mathrm{D}_A \oplus (V' + E)\,\mathrm{F}_z,$$

if V' is replaced by V in the previous equation.

From $A^* \supset A$ it follows that if

$$(4) \qquad f = f_0 + g_z + V'g_z \quad (g_z \in \mathrm{F}_z),$$

then

$$(4') \qquad \tilde A f = Af_0 + zg_z + \bar z V' g_z.$$

Formulas (1) and (4) will be called respectively the first and second Neumann formulas.

Let the deficiency indices of the operator A and its symmetric extension $\tilde A$ be (m, n) and $(m - p, n - p)$ where $m, n < \infty$. Then the second Neumann formula yields

$$\dim\ \mathrm{D}_{\tilde A} = p \pmod{\mathrm{D}_A}.$$

We now illustrate the above theory on the differential operator P which was introduced in Section 49. The equation

$$P^*g = zg$$

has the form

$$\frac{dg}{dt} + izg = 0.$$

Its formal solution is the function

$$(5) \qquad g(t) = Ce^{-izt}.$$

In the case with $-\infty < t < \infty$, this function belongs to L^2 only for $C = 0$. Hence, the deficiency indices of the differential operator P on the entire real axis equal (0, 0). In the case of the semi-axis ($0 \leqq t < \infty$) the deficiency indices are (0, 1), since the function (5) belongs to $L^2(0, \infty)$ for $\Im z < 0$ and does not belong to $L^2(0, \infty)$ for $\Im z > 0$. Finally in the case of the interval ($0 \leqq t \leqq 2\pi$) the deficiency indices are (1, 1), since the function (5) belongs to $L^2(0, 2\pi)$ for any z.

If we let in formula (4)

$$z = i, \quad g_z = e^t, \quad V'g_z = \theta g_{\bar{z}},$$

where $g_{\bar{z}} = e^{2\pi - t}$ and $\theta\,(\,|\,\theta\,| = 1)$ is a fixed constant and let $\arg\theta$ vary in the interval $[0, 2\pi]$, then we obtain all self-adjoint extensions $\tilde{P}$ of the operator P (for the interval $[0, 2\pi]$) in the form

$$\varphi(t) = \varphi_0(t) + \alpha\,(e^t + \theta e^{2\pi - t}),$$
$$\tilde{P}\varphi(t) = i\varphi_0'(t) + \alpha i\,(e^t - \theta e^{2\pi - t}).$$

Here $\varphi_0(0) = \varphi_0(2\pi) = 0$ and α is an arbitrary constant.

It is easy to verify that this result coincides with the result of Section 49; moreover the relation between the parameter here and the parameter there is given by

$$\theta = \frac{\beta - e^{2\pi}}{1 - e^{2\pi}\beta}.$$

81. Simple Symmetric Operators

A symmetric (respectively, isometric) operator A is *simple* if there does not exist a subspace invariant under A such that the restriction of A to this subspace is self-adjoint (respectively, unitary).

A symmetric operator A is simple if and only if its Cayley transform V is simple. This fact is a consequence of the following proposition.

THEOREM 1: *A subspace* G *reduces a symmetric operator A if and only if* G *reduces its Cayley transform V.*

Proof: Choose $h \in D_A$ arbitrarily. Then

$$h = (V - E)f, \quad Ah = (\bar{z}V - zE)f \quad (f \in D_V).$$

Assume that the subspace G reduces V and let P denote the operator of projection on G. Then

$$Pf \in D_V, \quad VPf = PVf.$$

Therefore,

$$Ph = (V - E)\,Pf \in \mathrm{D}_A$$

and

$$APh = A(V - E)Pf = (\bar{z}V - zE)Pf = P(\bar{z}V - zE)f = PAh.$$

Thus, one of the assertions of the theorem is proved.

We now prove the other assertion. Choose $f \in \mathrm{D}_V$ arbitrarily. Then

$$f = (A - \bar{z}E)h, \quad Vf = (A - zE)h \quad (h \in \mathrm{D}_A).$$

Assume that the subspace G reduces the operator A. Then

$$Pf = (A - \bar{z}E)\,Ph \in \mathrm{D}_V.$$

It remains to verify that

$$VPf = PVf.$$

Since

$$VPf = V(A - \bar{z}E)Ph = (A - zE)Ph = P(A - zE)h = PVf,$$

the theorem is proved.

If an isometric operator V is not simple, so that it has unitary restrictions, then there exists a *maximal unitary restriction* of V (which is an extension of every unitary restriction of V). Indeed, the maximal unitary restriction of the operator V is the restriction of V to the closed linear envelope G_0 of all the subspaces G invariant under V such that the restriction of V to G is unitary.

Analogously, there exists a *maximal self-adjoint restriction* of a non-simple symmetric operator.

LEMMA: *Let V be an isometric operator with equal deficiency indices, U_0 its maximal unitary restriction and U any unitary extension of the operator V. Then D_{U_0} is the orthogonal complement of the linear envelope* L *of the subspaces* $U^k(\mathrm{H} \ominus \mathrm{D}_V)$ $(\pm\, k = 0, 1, 2, \ldots)$.

Proof: The subspace $\mathrm{G} = \mathrm{H} \ominus \mathrm{L}$ is invariant under both U and U^{-1}. Therefore, by Theorem 3 of Section 42, the subspace G reduces the operator U. Let U' be the restriction of U to G. Since G is orthogonal to L, it is also orthogonal to $\mathrm{H} \ominus \mathrm{D}_V$. Hence,

$$\mathrm{G} \subset \mathrm{D}_V.$$

By similar reasoning,

$$\mathrm{G} \subset \Delta_V.$$

Hence, G reduces V and U' is a unitary restriction of V to G. Therefore,

$$\mathrm{G} \subset \mathrm{D}_{U_0}. \tag{1}$$

But, on the other hand,

$$\mathrm{D}_{U_0} \subset \mathrm{D}_V.$$

This implies that D_{U_0} is orthogonal to the subspace $\mathrm{H} \ominus \mathrm{D}_V$. Since

$$U^k\mathrm{D}_{U_0} = U_0^k\mathrm{D}_{U_0} = \mathrm{D}_{U_0},$$

$U^k\mathrm{D}_{U_0}$ is orthogonal to $\mathrm{H} \ominus \mathrm{D}_V$ for $k = 0, \pm 1, \pm 2, \ldots$. Hence, D_{U_0} is orthogonal to L, so that

$$\mathrm{D}_{U_0} \subset \mathrm{G}. \tag{2}$$

A comparison of (1) and (2) completes the proof of the lemma.

The following result is an immediate consequence of the lemma.

COROLLARY: *In a nonseparable space there exists no simple symmetric operator with equal and finite deficiency numbers.*

A further consequence of the lemma is the following.

THEOREM 2: *An isometric operator V with equal deficiency numbers is simple if and only if for each unitary extension U of V the linear envelope of the subspaces $U^k(\mathrm{H} \ominus \mathrm{D}_V)$ $(\pm k = 0, 1, 2, \ldots)$ is dense in* H.[3]

From this theorem and Theorem 1 of Section 69 it follows that every unitary (self-adjoint) extension of a simple isometric (symmetric) operator with deficiency indices (1, 1) has a simple spectrum.

Analogously, one proves the following general result.

THEOREM 3: *The multiplicity of the spectrum of a unitary (self-adjoint) extension of a simple isometric (symmetric) operator with finite deficiency indices (m, m) does not exceed m.*

82. The Structure of Maximal Operators

Let the space H be separable and let $\{e_k\}_1^\infty$ be any orthonormal basis for H. We define linear operators V_+ and V_- by the formulas

$$V_+e_k = e_{k+1} \qquad (k = 1, 2, 3, \ldots),$$
$$V_-e_k = e_{k-1} \qquad (k = 2, 3, 4, \ldots).$$

It is evident that V_+ and V_- are isometric operators with deficiency indices (0, 1) and (1, 0) respectively.

Since we wish to use the Cayley transforms A_+ and A_- of the operators V_+ and V_-, we must show that the manifolds $\Delta_{V+}(1)$ and $\Delta_{V-}(1)$ are dense in H. We present the proof only for $\Delta_{V+}(1)$. Letting

$$f = e_k + \frac{r-1}{r} e_{k+1} + \ldots + \frac{1}{r} e_{k+r-1} \quad (k = 1, 2, 3, \ldots),$$

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[3] Translator's Note: In the German edition, there is the following additional statement: Such an operator V is simple if there exists a unitary operator U with the given properties.

we find that

$$(V_+ - E)f = \left(e_{k+1} + \frac{r-1}{r} e_{k+2} + \dots + \frac{1}{r} e_{k+r}\right)$$
$$- \left(e_k + \frac{r-1}{r} e_{k+1} + \dots + \frac{1}{r} e_{k+r-1}\right) =$$
$$= \frac{1}{r}(e_{k+1} + e_{k+2} + \dots + e_{k+r}) - e_k.$$

But

$$\left\| \frac{1}{r} (e_{k+1} + e_{k+2} + \dots + e_{k+r}) \right\|^2 = \frac{1}{r} ;$$

whence we conclude that the basis vectors $e_k (k = 1, 2, 3, \dots)$ are limits of vectors of $\Delta_{V+}(1)$. Therefore, the manifold $\Delta_{V+}(1)$ is dense in H.

The symmetric operators A_+ and A_- are maximal with deficiency indices (0, 1) and (1, 0) respectively.

THEOREM 1: *The operators A_+ and A_- are irreducible.*

For the proof it is sufficient to show (cf. Section 81) that the operators V_+ and V_- are irreducible. We proceed to prove that V_+ is irreducible. Assume that the subspace F (and, hence, its orthogonal complement G) reduces V_+ and let $V_+\mathrm{F} = \mathrm{F}_+$ and $V_+\mathrm{G} = \mathrm{G}_+$ so that

$$\mathrm{F}_+ \subset \mathrm{F}, \quad \mathrm{G}_+ \subset \mathrm{G}.$$

In the last two relations, simultaneous equality is not possible, for $\mathrm{F}_+ = \mathrm{F}$ and $\mathrm{G}_+ = \mathrm{G}$ would imply that $V_+\mathrm{H} = \mathrm{H}$. Let, for example, $\mathrm{F} \ominus \mathrm{F}_+ \neq 0$ and

$$f \in \mathrm{F} \ominus \mathrm{F}_+ \qquad (f \neq 0).$$

Since $f \in \mathrm{F}$, f is orthogonal to G. Hence,

$$(1) \qquad f \perp \mathrm{F}_+ \oplus \mathrm{G}_+.$$

From (1) it follows that $f \perp e_k$ $(k = 2, 3, \dots)$. Therefore, $f = \alpha e_1 (\alpha \neq 0)$, whence $e_1 \in \mathrm{F} \ominus \mathrm{F}_+$ and $e_1 \in \mathrm{F}$.

Since F reduces V_+, F contains e_k $(k = 2, 3, \dots)$ as well as e_1. So $\mathrm{F} = \mathrm{H}$ and the assertion concerning A_+ is proved. The proof for A_- is quite similar.

Thus, the operators A_+ and A_- are irreducible and, hence, are simple symmetric operators. The importance of these operators is indicated by the following theorem.

THEOREM 2: (J. von Neumann) *If a simple symmetric operator A in the space* H *has deficiency indices* (0, 1) (*respectively*, (1, 0)), *then the space*

H *is separable and the operator* A *is isomorphic to the operator* A_+ (*respectively,* A_-).

Proof: Let us assume for definiteness that the deficiency indices of the operator A are (0, 1). Let V be the Cayley transform of the operator A and let $e_1 \perp \Delta_V (\|e_1\|=1)$. We form the closed linear envelope M of the vectors $V^k e_1 = e_{k+1}$ $(k = 0, 1, 2, \ldots)$; M is a subspace with the orthonormal basis $\{e_k\}_1^\infty$.

Since M is invariant with respect to V and V^{-1}, M reduces V and, hence, H $\ominus$ M also reduces V. The restriction of V to M is evidently isomorphic to V_+ and, hence, has deficiency indices (0, 1). The restriction of V to H $\ominus$ M must be unitary since otherwise at least one of the deficiency indices of V would exceed the corresponding deficiency index of the operator V_+ and, therefore, of the operator A.

On the other hand, a simple unitary operator V has no proper unitary restriction. Thus, H $\ominus$ M $= 0$ and the operator V is isomorphic to V_+. Hence, its Cayley transform A is isomorphic to A_+.

With the aid of the operator A_+ (respectively, A_-) it is possible to construct a maximal operator with the deficiency indices $(0, \mathfrak{n})$, (respectively, $(\mathfrak{m}, 0)$). For this purpose, it is sufficient to form the direct sum of separable Hilbert spaces H_α, where α runs through a set of cardinality $\mathfrak{n}$ (respectively, $\mathfrak{m}$), and in each of these to realize the operator A_+ (respectively, A_-).

As J. von Neumann first showed, in this way it is possible to get any simple maximal operator, i.e., the following theorem holds.

THEOREM 3: *A simple symmetric operator A with deficiency indices* $(0, \mathfrak{n})$ (*or* $(\mathfrak{m}, 0)$) *can be decomposed into a direct sum of operators* A_+ (*respectively,* A_-):

$$A = \sum_{\alpha \in M} \oplus A_+^\alpha \quad ,$$

where cardinality (M) $= \mathfrak{N}$; *or, correspondingly*

$$A = \sum_{\alpha \in M} \oplus A_-^\alpha \quad ,$$

where cardinality (M) $= \mathfrak{m}$.

Proof: Let V be the Cayley transform of the operator A, $D_V = H$ and $H \ominus D_V = M_1$ ($\dim M_1 = \mathfrak{n}$). We let

$$V^k M_1 = M_{k+1} \qquad (k = 0, 1, 2, \ldots)$$

and denote by M the closure of the linear envelope of the spaces $M_k (k = 1, 2, 3, \ldots)$.

In the same way as in Theorem 2, we get

$$M_i \perp M_k \qquad (i \neq k;\ i, k = 1, 2, 3, \ldots),$$
$$M = H.$$

Let $e_k^{(a)}$ $(k = 1, 2, 3, \ldots)$ be a complete orthonormal system in M_k and let $\mathrm{H}^{(a)}$ be the closure of the linear envelope of the vectors $e_1^{(a)}, e_2^{(a)}, e_3^{(a)}, \ldots$. The spaces $\mathrm{H}^{(a)}$ and $\mathrm{H}^{(a')}$ are orthogonal for $a \neq a'$ and the direct sum of all these subspaces is H.

It is evident that each of the subspaces $\mathrm{H}^{(a)}$ reduces V and that the restriction of V to $\mathrm{H}^{(a)}$ is isomorphic to V_+. The proof of the theorem is completed by means of the Cayley transform.

Theorem 3 provides the answer to the question about the structure of simple maximal operators.

The operators A_+ and A_- (respectively, V_+ and V_-) will be called *elementary maximal operators.*

With the aid of these elementary maximal operators it is possible to construct an operator with given deficiency indices $(\mathfrak{m}, \mathfrak{n})$. To accomplish this, one need only construct, by the described above method, operators with deficiency indices $(\mathfrak{m}, 0)$ and $(0, \mathfrak{n})$ and then form their direct sum. However, an arbitrary simple operator with deficiency indices $(\mathfrak{m}, \mathfrak{n})$ in general can not be constructed from elementary maximal operators.

As an illustration of the proposition of the present section we consider again the differential operator P on the positive real semi-axis. This operator has deficiency indices $(0, 1)$. In order to study the structure of the operator P we pass to its Cayley transform

$$V = (P - iE)(P + iE)^{-1}$$

and form the powers $V^k g$, where $g = e^{-t}$ is a deficiency element of P for the point $-i$. Note that e^{-t} is the Tchebysheff-Laguerre function of order zero with (cf. Section 11) the argument doubled:

$$e^{-t} = \psi_0(2t).$$

We shall show that

$$V\psi_k(2t) = \psi_{k+1}(2t) \qquad (k = 0, 1, 2, \ldots).$$

We represent $\psi_k(2t)$ in the form

$$\psi_k(2t) = if'(t) + if(t) \equiv (P + iE)f,$$

where $f(0) = 0$. It follows that

$$f(t) = -ie^{-t}\int_0^t \psi_k(2s)\, e^s\, ds,$$

and, therefore,

$$V\psi_k(2t) = (P - iE)f = -2e^{-t}\int_0^t \psi_k(2s)\, e^s\, ds + \psi_k(2t).$$

Thus, we must establish the identity

$$\psi_{k+1}(2t) = \psi_k(2t) - 2e^{-t}\int_0^t \psi_k(2s)\, e^s\, ds,$$

and this presents no difficulty. Since the Tchebysheff-Laguerre functions form a complete orthonormal system in $L^2(0, \infty)$, the operator V is isomorphic to V_+ and hence P is isomorphic to A_+.

Thus, Theorem 2 means that every simple operator with deficiency indices (0, 1) or (1, 0) is isomorphic to the differential operator P on the semi-axis $(0, \infty)$ or, correspondingly, $(-\infty, 0)$.

The operator P of differentiation on the axis $(-\infty, \infty)$ is evidently isomorphic to some self-adjoint extension of the operator $A_+ \oplus A_-$. Thus, (cf. Section 77) we can prove by different means the unitary equivalence of the differential operator P and the operator of multiplication by the independent variable.

83. Spectra of Self-Adjoint Extensions of Symmetric Operators

According to the general definition given in Section 43, the spectrum of a symmetric, but not self-adjoint, operator A contains the complement of the set of all points of regular type of A. However, the spectrum is not exhausted by this complement. For example, at least one of the open half-planes $\Im z > 0$ and $\Im z < 0$ belongs to the spectrum. Nevertheless, this complement forms a particularly important part of the spectrum.

We shall call the complement of the set of all points of regular type of a symmetric operator the *spectral kernel* of this operator. In order to classify the points in the spectral kernel of a symmetric operator A, we first denote by A_λ the restriction of the operator A to the subspace $\mathrm{H} \ominus \mathrm{G}_\lambda$, where G_λ is the eigenmanifold of A corresponding to λ if λ is an eigenvalue of A and G_λ is the null manifold otherwise.

In the spectral kernel we single out a particular subset. This is the set of all eigenvalues of the operator (it is empty if the operator is simple). We call this set the *discrete part* of the spectral kernel.

Turning now to the characterization of the remaining part of the spectral kernel of the operator A, we remark that for each λ the operator $A_\lambda - \lambda E$ has an inverse. The set of all λ for which the operator $(A_\lambda - \lambda E)^{-1}$ is unbounded obviously belongs to the spectral kernel. We call this set the continuous part of the spectral kernel. Thus, every point of the spectral kernel belongs either to the discrete part or to the continuous part or to both simultaneously.

For a self-adjoint operator the concepts of regular point and point of regular type coincide. Therefore, the spectral kernel of a self-adjoint operator coincides with the spectrum of this operator. Hence, the spectral kernel of a self-adjoint operator cannot be the empty set. For an arbitrary symmetric operator, such an assertion would be incorrect.

If $\tilde{A}$ is a symmetric (in particular, maximal or self-adjoint) extension of the operator A, then the spectral kernel of $\tilde{A}$ contains the spectral kernel of A. Moreover, each of the parts (discrete and continuous) of the spectral kernel of $\tilde{A}$ contains the corresponding part of the spectral kernel of A.

We now consider a special case, for which the continuous part of the spectral kernel of the operator A is invariant for symmetric extensions of this operator. This happens if the deficiency numbers of the operator A are finite. In this case, by the second Neumann formula (cf. Section 80), the manifold $(\tilde{A}_\lambda - \lambda E)\mathrm{D}_{\tilde{A}}$, where $\tilde{A}$ is any symmetric extension of the operator A, contains the manifold $(A_\lambda - \lambda E)\,\mathrm{D}_A$. However, the difference in dimensions is finite (i.e., the dimension of the first manifold modulo the second is finite). Therefore, the operator $(\tilde{A}_\lambda - \lambda E)^{-1}$ is bounded along with the operator $(A_\lambda - \lambda E)^{-1}$.

From these remarks follow the simple theorem.

THEOREM 1: *All self-adjoint extensions of an operator with equal and finite deficiency indices have the same continuous spectrum.*

Concerning the set of eigenvalues in the spectral kernel we have

THEOREM 2: *For an arbitrary extension of an operator with finite deficiency indices* (m, n) *to a self-adjoint operator, the multiplicity of an arbitrary eigenvalue is increased not more than* m *(in particular, the multiplicity of a new eigenvalue is not larger than* m*).*

Proof: Let $\tilde{A}$ be a self-adjoint extension of the operator A and let λ be an eigenvalue of multiplicity p of A. We assume that the multiplicity of λ as eigenvalue of the operator $\tilde{A}$ is equal to $p + q$ with $q > m$, contrary to the assertion of the theorem. We select a linear independent system of solutions $f_1, f_2, \ldots, f_p, f_{p+1}, \ldots, f_{p+q}$ of the equation $\tilde{A}f - \lambda f = 0$, so that $f_k \in \mathrm{D}_A$ for $k \leqq p$. Since the dimension of $\mathrm{D}_{\tilde{A}}$ modulo D_A is equal to m, there exists constants α_k such that

$$\alpha_1 f_{p+1} + \alpha_2 f_{p+2} + \ldots + \alpha_q f_{p+q} \in \mathrm{D}_A.$$

This equation implies that the multiplicity of λ as eigenvalue of the operator A is larger than p. Since this is a contradiction, the theorem is proved.

The following theorem is in some sense a converse of Theorem 2.

THEOREM 3: *If* λ *is a real point of regular type of a symmetric opera-*

tor A with finite deficiency indices (m, m), then there exists a self-adjoint extension $\tilde{A}$ of A for which the number λ is an eigenvalue of multiplicity m.

Proof: Let $\mathfrak{N}_\lambda$ denote the linear manifold of all solutions of the equations

$$A^*g - \lambda g = 0.$$

By the theorem on the invariance of the deficiency numbers in the field of regularity (cf. Section 78), the dimension of $\mathfrak{N}_\lambda$ is m.

The domain D_A of the operator A and the linear manifold $\mathfrak{N}_\lambda$ are linearly independent since, otherwise, the number λ would be an eigenvalue of A. Let

$$\mathrm{D} = \mathrm{D}_A \oplus \mathfrak{N}_\lambda. \tag{1}$$

Let $\tilde{A}$ be the operator which coincides with A^* on $\mathrm{D} = \mathrm{D}_{\tilde{A}}$. Then λ is an eigenvalue of A of multiplicity m.

We show now that the operator $\tilde{A}$ is self-adjoint. For this it is sufficient to prove that $\tilde{A}$ is symmetric, since it follows from (1) that

$$\dim \mathrm{D}_{\tilde{A}} = m \qquad (\operatorname{mod} \mathrm{D}_A).$$

If f and g are arbitrary elements of $\mathrm{D}_{\tilde{A}}$ and

$$\begin{aligned} f &= f_1 + f_2 \qquad (f_1 \in \mathrm{D}_A,\ f_2 \in \mathfrak{N}_\lambda), \\ g &= g_1 + g_2 \qquad (g_1 \in \mathrm{D}_A,\ g_2 \in \mathfrak{N}_\lambda), \end{aligned}$$

then

$$\begin{aligned} (\tilde{A}f, g) &= (Af_1, g_1) + (A^*f_2, g_1) + (Af_1, g_2) + (A^*f_2, g_2) = \\ &= (Af_1, g_1) + \lambda(f_2, g_1) + \lambda(f_1, g_2) + \lambda(f_2, g_2), \end{aligned}$$

and

$$\begin{aligned} (f, \tilde{A}g) &= (f_1, Ag_1) + (f_1, A^*g_2) + (f_2, Ag_1) + (f_2, A^*g_2) = \\ &= (f_1, Ag_1) + \lambda(f_1, g_2) + \lambda(f_2, g_1) + \lambda(f_2, g_2). \end{aligned}$$

It follows that $\tilde{A}$ is symmetric.

In conclusion we give another theorem on the number of solutions of the equation

$$A^*g - \lambda g = 0$$

for real λ.

THEOREM 4: *If A is a symmetric operator with finite deficiency indices (m, m) and λ is a real number not belonging to the point spectrum of the operator A, then the number $m(\lambda)$ of the solutions of the equations*

$$A^*g - \lambda g = 0 \tag{2}$$

does not exceed the deficiency number m.

Proof: We define the domain $\mathrm{D}_{\tilde{A}}$ by formula (1), where $\mathfrak{N}_\lambda$ is the linear manifold of the solutions of equation (2). Then $\tilde{A} \subset A^*$. From the proof of the preceding theorem it follows that $\tilde{A}$ is a symmetric extension of A, so that

$$m(\lambda) = \dim D_{\tilde{A}} \leqq m \qquad (\mathrm{mod}\ D_A).$$

We shall refer to this last theorem in Appendix II.

84. The Formula of Krein for the Resolvent of the Self-Adjoint Extensions of a Symmetric Operator

In this section we consider symmetric operators with equal and finite deficiency indices.

Let A_1 and A_2 be two self-adjoint extensions of an operator A with finite deficiency indices (m, m):

$$A_1 \supset A, \qquad A_2 \supset A.$$

It is natural to call an operator C which satisfies the conditions

$$(1) \qquad A_1 \supset C, \qquad A_2 \supset C$$

a common part of the operators A_1 and A_2. It is evident that there exists an operator C which satisfies conditions (1) and which is an extension of every common part of A_1 and A_2. We call such an operator the *maximal common part* of A_1 and A_2. The maximal common part either is an extension of A or coincides with A and in the latter case we call the extensions A_1 and A_2 *relatively prime.* Two extensions A_1 and A_2 are relatively prime if and only if the conditions

$$(2) \qquad h \in D_{A_1}, \quad h \in D_{A_2}$$

imply that $h \in D_A$.

If the maximal number of vectors which are linearly independent modulo D_A and which satisfy conditions (2) is equal to $p (0 \leqq p \leqq m)$, then the maximal common part A_0 of the operators A_1 and A_2 has deficiency indices $(m - p, m - p)$. In this case the operators A_1 and A_2 can be considered as relatively prime self-adjoint extensions of A_0.

The problem of the present section is the derivation of a formula which relates the resolvents of two self-adjoint extensions of an operator A. Let $\mathring{B}$ be a fixed self-adjoint extension, B an arbitrary self-adjoint extension, and let $\mathring{R}_z$ and R_z be their resolvents. Further, let λ be any common regular point of the operators $\mathring{B}$ and B (in particular, λ can be an arbitrary nonreal number).

In order not to exclude the case when $\mathring{B}$ and B are not relatively prime extensions of A, we shall consider them as relatively prime extensions of their maximal common part A_0 which has deficiency indices (r, r) where $0 < r \leqq m$.

We now let $\mathfrak{M}_\lambda = \Delta_{A_0}(\lambda)$ and $\mathfrak{N}_\lambda = \mathrm{H} \ominus \mathfrak{M}_\lambda$. Then the difference of the resolvents satisfies the formula

$$(3) \qquad (\overset{\circ}{R}_\lambda - R_\lambda) f \begin{cases} = 0 & \text{for } f \in \mathfrak{M}_\lambda, \\ \in \mathfrak{N}_{\bar\lambda} & \text{for } f \in \mathfrak{N}_\lambda. \end{cases}$$

This follows from the fact that, for each $h \in \mathfrak{M}_{\bar\lambda}$,

$$(\{\overset{\circ}{R}_\lambda - R_\lambda\} f, h) = (f, \{\overset{\circ}{R}_\lambda - R_\lambda\}^* h) = (f, \{\overset{\circ}{R}_{\bar\lambda} - R_{\bar\lambda}\} h) = (f, 0) = 0.$$

Now we choose any r linearly independent vectors $g_1(\bar\lambda), g_2(\bar\lambda), \ldots, g_r(\bar\lambda)$ from $\mathfrak{N}_\lambda$ and r linearly independent vectors $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$ from $\mathfrak{N}_{\bar\lambda}$. It follows from (3) that, for each $f \in \mathrm{H}$,

$$(4) \qquad (\overset{\circ}{R}_\lambda - R_\lambda) f = \sum_{k=1}^{r} c_k g_k(\lambda).$$

By (4), the constants c_k are linear functionals of f. Hence, there exist vectors $h_k(\lambda)$ such that

$$c_k = (f, h_k(\lambda)) \qquad (k = 1, 2, \ldots, r).$$

Since the vectors $g_1(\lambda), g_2(\lambda), \ldots, g_r(\lambda)$ are linearly independent, it follows from (3) and (4) that, for each $f \perp \mathfrak{N}_\lambda$,

$$(f, h_k(\lambda)) = 0 \qquad (k = 1, 2, \ldots, r).$$

Therefore,

$$h_k(\lambda) \in \mathfrak{N}_\lambda \qquad (k = 1, 2, \ldots, r),$$

so that each $h_k(\lambda)$ has a representation of the form

$$(5) \qquad h_k(\lambda) = \sum_{i=1}^{r} \overline{p_{ik}(\lambda)} g_i(\bar\lambda) \qquad (k = 1, 2, \ldots, r),$$

and (4) can be expressed by

$$(6) \qquad (\overset{\circ}{R}_\lambda - R_\lambda) f = \sum_{i,k=1}^{r} p_{ik}(\lambda) (f, g_i(\bar\lambda)) g_k(\lambda).$$

We remark that the matrix function $(p_{ik}(\lambda)) = \mathfrak{P}(\lambda)$, which is defined on the set of all common regular points of the operators $\overset{\circ}{B}$ and B, is nonsingular. Indeed, if the determinant of $(p_{ik}(\lambda_0))$ were zero then, by (5), the vectors $h_k(\lambda_0)$ $(k = 1, 2, \ldots, r)$ would be linearly dependent, and this would imply the existence of a vector $h \neq 0$ such that

$$h \perp h_k(\lambda_0), \quad h \in \mathfrak{N}_{\lambda_0} \quad (k = 1, 2, \ldots, r).$$

Then it would follow from (4) that

$$(\overset{\circ}{R}_{\lambda_0} - R_{\lambda_0}) h = 0.$$

This would contradict the fact that $\overset{\circ}{B}$ and B are relatively prime extensions of the operator A_0.

In (6) we omit the element f and consider the expressions $(\cdot, g_i(\bar{\lambda}))g_k(\lambda)$ $(i, k = 1, 2, \ldots, r)$ as operators in order to obtain, for each common regular point λ of the operators $\overset{\circ}{B}$ and B, the formula

$$R_\lambda = \overset{\circ}{R}_\lambda - \sum_{i,k=1}^{r} p_{ik}(\lambda)\,(\cdot, g_i(\bar{\lambda}))\, g_k(\lambda). \tag{7}$$

Until now the choice of the vector functions $g_k(\lambda)$ and $g_i(\bar{\lambda})$ $(i, k = 1, 2, \ldots, r)$ have been arbitrary. At the same time the left member and, hence, also the right member of formula (6) is a regular analytic vector function of λ. We show next that $g_k(\lambda)$ $(k = 1, 2, \ldots, r)$ can be defined as a regular analytic function of λ and then we obtain a formula for the matrix function $\mathfrak{P}(\lambda)$ which corresponds to this choice.

With this purpose, we choose an arbitrary fixed value λ_0 and introduce the operator

$$U_{\lambda\lambda_0} = \frac{\overset{\circ}{B} - \lambda_0 E}{\overset{\circ}{B} - \lambda E} = E + (\lambda - \lambda_0)\, \overset{\circ}{R}_\lambda$$

with domain

$$(\overset{\circ}{B} - \lambda E)\, \mathrm{D}_{\overset{\circ}{B}} = \mathrm{H}$$

and range

$$(\overset{\circ}{B} - \lambda_0 E)\, \mathrm{D}_{\overset{\circ}{B}} = \mathrm{H}.$$

The operator $U_{\lambda\lambda_0}$ is defined by the formulas

$$\left.\begin{aligned} (\overset{\circ}{B} - \lambda E) f &= h, \\ (\overset{\circ}{B} - \lambda_0 E) f &= U_{\lambda\lambda_0} h, \end{aligned}\right\} \quad (f \in \mathrm{D}_{\overset{\circ}{B}}),$$

from which it follows that $U_{\lambda\lambda_0}$ determines a one-to-one correspondence of H to H.

In the special case with $\lambda = \bar{\lambda}_0$, the operator $U_{\lambda\lambda_0}$ becomes the Cayley transform of $\overset{\circ}{B}$ and it maps the deficiency subspace $\mathfrak{N}_{\bar{\lambda}_0}$ of A_0 into its deficiency subspace $\mathfrak{N}_{\lambda_0}$. We show that, in general,

$$U_{\lambda\lambda_0}\, \mathfrak{N}_{\bar{\lambda}_0} = \mathfrak{N}_{\bar{\lambda}}.$$

Let $g_1(\lambda_0), g_2(\lambda_0), \ldots, g_r(\lambda_0)$ be an arbitrary basis for $\mathfrak{N}_{\bar{\lambda}_0}$ (the vectors $g_k(\lambda_0)$ are not necessarily orthogonal or normalized). Now,

$$U_{\lambda\lambda_0} g_k(\lambda_0) \in \mathfrak{N}_{\bar{\lambda}}$$

since

$$\begin{aligned}A_0^* U_{\lambda\lambda_0} g_k(\lambda_0) &= A_0^* \{E + (\lambda - \lambda_0)\mathring{R}_\lambda\} g_k(\lambda_0) = \\ &= \lambda_0 g_k(\lambda_0) + (\lambda - \lambda_0)\mathring{B}\mathring{R}_\lambda g_k(\lambda_0) = \\ &= \lambda_0 g_k(\lambda_0) + (\lambda - \lambda_0)(E + \lambda\mathring{R}_\lambda) g_k(\lambda_0) = \\ &= \lambda \{E + (\lambda - \lambda_0)\mathring{R}_\lambda\} g_k(\lambda_0) \\ &= \lambda\, U_{\lambda\lambda_0} g_k(\lambda_0).\end{aligned}$$

Since the operator $U_{\lambda\lambda_0}$ is one-to-one, the vectors $U_{\lambda\lambda_0}\, g_k(\lambda_0)$ form a basis for $\mathfrak{N}_{\bar\lambda}$. Hence, we may define the vectors $g_k(\lambda)$ at each regular point λ of the operator $\mathring{B}$ by the formulas

$$g_k(\lambda) = U_{\lambda\lambda_0} g_k(\lambda_0) = g_k(\lambda_0) + (\lambda - \lambda_0)\mathring{R}_\lambda g_k(\lambda_0) \quad (k = 1, 2, \ldots, \mathrm{r}).$$

With the aid of the functional equation of the resolvent, it is easy to verify that

$$g_k(\mu) = U_{\mu\lambda} g_k(\lambda) = g_k(\lambda) + (\mu - \lambda)\mathring{R}_\mu g_k(\lambda) \tag{8}$$

for any two regular points λ and μ of the operator $\mathring{B}$.

The value of the matrix function $\mathfrak{P}(\lambda)$ for any common regular point λ of $\mathring{B}$ and B is determined by its value $\mathfrak{P}(\lambda_0)$. In order to find the corresponding formula we make use of the functional equation of the resolvent:

$$R_\lambda = R_{\lambda_0} + (\lambda - \lambda_0) R_\lambda R_{\lambda_0}. \tag{9}$$

On the other hand, by (7)

$$\begin{aligned}R_\lambda &= \mathring{R}_\lambda - \sum_{i,k=1}^{r} p_{ik}(\lambda)(\cdot, g_i(\bar\lambda))\, g_k(\lambda), \\ R_{\lambda_0} &= \mathring{R}_{\lambda_0} - \sum_{i,k=1}^{r} p_{ik}(\lambda_0)(\cdot, g_i(\bar\lambda_0))\, g_k(\lambda_0).\end{aligned} \tag{10}$$

Substitute (10) into (9) and use the functional equation of the resolvent $\mathring{R}_\lambda$ to get

$$\begin{aligned}-\sum_{i,k=1}^{r} p_{ik}(\lambda)(\cdot, g_i(\bar\lambda))\, g_k(\lambda) &= -\sum_{i,k=1}^{r} p_{ik}(\lambda_0)(\cdot, g_i(\bar\lambda_0))\, g_k(\lambda_0) \\ &\quad - (\lambda - \lambda_0)\sum_{i,k=1}^{r} p_{ik}(\lambda_0)(\cdot, g_i(\bar\lambda_0))\,\mathring{R}_\lambda g_k(\lambda_0) \\ &\quad - (\lambda - \lambda_0)\sum_{i,k=1}^{r} p_{ik}(\lambda)(\mathring{R}_{\lambda_0}\cdot, g_i(\bar\lambda))\, g_k(\lambda) \\ &\quad + (\lambda - \lambda_0)\sum_{i,k,j,s=1}^{r} p_{ik}(\lambda)(g_s(\lambda_0), g_i(\bar\lambda))\, p_{js}(\lambda_0)(\cdot, g_j(\bar\lambda_0))\, g_k(\lambda).\end{aligned} \tag{11}$$

By means of (8), we express the sum of the second and third summands on the right in the form

$$\sum_{i,k=1}^{r} p_{ik}(\lambda_0)(\cdot\,, g_i(\bar{\lambda}_0))\{g_k(\lambda_0) - g_k(\lambda)\} + \sum_{i,k=1}^{r} p_{ik}(\lambda)(\cdot\,, g_i(\bar{\lambda}_0) - g_i(\bar{\lambda}))\, g_k(\lambda).$$

With this substitution, (11) yields

$$-\sum_{i,k=1}^{r} p_{ik}(\lambda_0)(\cdot\,, g_i(\bar{\lambda}_0))\, g_k(\lambda) + \sum_{i,k=1}^{r} p_{ik}(\lambda)(\cdot\,, g_i(\bar{\lambda}_0))\, g_k(\lambda) +$$

$$+ (\lambda - \lambda_0) \sum_{i,k,j,s=1}^{r} p_{ik}(\lambda)(g_s(\lambda_0), g_i(\bar{\lambda}))\, p_{js}(\lambda_0)(\cdot\,, g_j(\bar{\lambda}_0))\, g_k(\lambda) = 0.$$

Since the vectors $g_k(\lambda)$ are linear independent,

$$-\sum_{i=1}^{r} p_{ik}(\lambda_0)(\cdot\,, g_i(\bar{\lambda}_0)) + \sum_{i=1}^{r} p_{ik}(\lambda)(\cdot\,, g_i(\bar{\lambda}_0)) +$$

$$+ (\lambda - \lambda_0) \sum_{i,j,s=1}^{r} p_{ik}(\lambda)(g_s(\lambda_0), g_i(\bar{\lambda}))\, p_{js}(\lambda_0)(\cdot\,, g_j(\lambda_0)) = 0,$$

and, further, since the vectors $g_i(\bar{\lambda}_0)$ are linearly independent,

$$-p_{ik}(\lambda_0) + p_{ik}(\lambda) + (\lambda - \lambda_0)\sum_{j,s=1}^{r} p_{is}(\lambda_0)(g_s(\lambda_0), g_j(\bar{\lambda}))\, p_{jk}(\lambda) = 0$$

or, in matrix form,

$$\mathfrak{P}(\lambda) - \mathfrak{P}(\lambda_0) + (\lambda - \lambda_0)\,\mathfrak{P}(\lambda_0)\big((g_s(\lambda_0), g_j(\bar{\lambda}))\big)\,\mathfrak{P}(\lambda) = 0.$$

Multiply this equation on the right by $\mathfrak{P}^{-1}(\lambda)$ and on the left by $\mathfrak{P}^{-1}(\lambda_0)$ to obtain finally the relations

$$\mathfrak{Q}(\lambda) = \mathfrak{Q}(\lambda_0) + (\lambda - \lambda_0)\big((g_s(\lambda_0), g_j(\bar{\lambda}))\big)_{s,j=1}^{r}\ , \tag{12}$$

where $\mathfrak{Q}(\lambda) = \mathfrak{P}^{-1}(\lambda)$. It is not difficult to verify by means of (12) that, for any two common regular points λ and μ of the operators $\mathring{B}$ and B,

$$\mathfrak{Q}(\lambda) = \mathfrak{Q}(\mu) + (\lambda - \mu)\big((g_s(\mu), g_j(\bar{\lambda}))\big)_{s,j=1}^{r}\ .$$

85. Semi-Bounded Operators

A symmetric operator A is *bounded below* if

$$(Af, f) \geqq m\,\|f\|^2 \qquad (m > -\infty) \tag{1}$$

for each $f \in D_A$, and *bounded above* if

$$(Af, f) \leqq M\,\|f\|^2 \qquad (M < \infty) \tag{2}$$

for each $f \in D_A$. We shall assume that the numbers m and M in the inequalities (1) and (2) cannot be improved. Thus,

$$m = \inf_{f \in D_A} \frac{(Af,f)}{(f,f)}, \qquad M = \sup_{f \in D_A} \frac{(Af,f)}{(f,f)},$$

and we call m the *lower bound* and M the *upper bound* of A.

A special case of an operator semi-bounded from below is the positive operator introduced in Section 41. The study of semi-bounded operators can be reduced to the study of positive operators. Indeed, the operator A has the lower bound $m > -\infty$ if and only if the operator $A - mE$ is positive.

If A is a positive operator, then the negative semi-axis belongs to its field of regularity, since the inequality

$$(Af,f) \geqq 0$$

implies, for negative λ,

$$\|(A - \lambda E)f\|^2 = \|Af\|^2 - 2\lambda(Af,f) + \lambda^2\|f\|^2 \geqq \lambda^2\|f\|^2$$

and, hence,

$$\|(A - \lambda E)f\| \geqq |\lambda| \cdot \|f\|.$$

Therefore, by proposition 1° of Section 78, a semi-bounded operator has equal deficiency numbers and has self-adjoint extensions. If the negative semi-axis belongs to the field of regularity of a symmetric operator A then it is not necessarily true that A is positive,[4] unless A is self-adjoint. In the latter case, for each $f \in D_A$,

$$(Af,f) = \int_{-\infty}^{\infty} t\,d(E_t f,f) = \int_{0}^{\infty} t\,d(E_t f,f) \geqq 0.$$

The square of any self-adjoint operator is a positive operator. Conversely, any positive self-adjoint operator A can be represented in the form of the square of some self-adjoint operator B. Indeed, if

$$A = \int_0^\infty t\,dE_t,$$

then it is possible, for example, to let

$$B = \int_0^\infty \sqrt{t}\,dE_t.$$

---

[4] For example, for the differential operator on a finite interval, the field of regularity is the entire plane, but this operator is not semi-bounded.

If a semi-bounded operator has finite deficiency indices, then all of its self-adjoint extensions are also semi-bounded. Moreover, the following theorem holds.

THEOREM 1: *If A is a positive operator with the finite deficiency indices (m, m), then each of its self-adjoint extensions has only a finite number of negative eigenvalues, and the sum of the multiplicities of these eigenvalues does not exceed m.*

PROOF: Let $\tilde{A}$ be some self-adjoint extension of the operator A and

$$\tilde{A} = \int_{-\infty}^{\infty} t\, dE_t.$$

Further, let Δ_ε denote the interval $(-\infty, -\varepsilon)$. To establish the theorem it is sufficient to prove that, for each $\varepsilon > 0$, the dimension of the subspace $E(\Delta_\varepsilon)$H does not exceed the deficiency number m.

We assume that for some $\varepsilon > 0$

$$\dim E(\Delta_\varepsilon)\, \mathrm{H} > m. \tag{3}$$

Since

$$\dim \mathrm{D}_{\tilde{A}} = m \qquad (\mathrm{mod}\ \mathrm{D}_A),$$

there exists a vector f_0 which is both in D_A and in the subspace $E(\Delta_\varepsilon)\,\mathrm{H}$. Then

$$(Af_0, f_0) = (\tilde{A}f_0, f_0) = \int_{-\infty}^{-\varepsilon} t\, d(E_t f_0, f_0) < 0.$$

Since this contradicts the fact that A is positive, the theorem is proved.

We present another theorem about self-adjoint extensions of semi-bounded operators with arbitrary deficiency indices.

THEOREM 2: *A semi-bounded operator A with lower bound m has a self-adjoint extension $\tilde{A}$ with lower bound not smaller than an arbitrarily pre-assigned number $m' < m$.*

Proof: First, we prove the theorem for $m = 1$ and $m' = 0$. In this case,

$$(f, f) \leqq (Af, f) \leqq \|Af\| \cdot \|f\|,$$

so that

$$\|Af\| \geqq \|f\|$$

and

$$\|Af - Ag\| \geqq \|f - g\|.$$

It follows that A^{-1} exists as a bounded operator defined on the subspace Δ_A. Repeating the argument of item 2° of Section 78, it is easy to see that

$$\mathfrak{N}_0 = \mathrm{H} \ominus \Delta_A$$

is the eigenmanifold of the operator A^* which belongs to the eigenvalue $\lambda = 0$. The subspace $\mathfrak{N}_0$ and the manifold D_A are linearly independent since if an element g belongs to both $\mathfrak{N}_0$ and D_A, then $Ag = 0$, which implies that $g = 0$.

We define an extension $\tilde{A}$ of the operator A on the domain

$$\mathrm{D}_{\tilde{A}} = \mathrm{D}_A \oplus \mathfrak{N}_0$$

by the formula

$$\tilde{A}h = Af$$

with

$$h = f + g, \quad f \in \mathrm{D}_A, \quad g \in \mathfrak{N}_0.$$

It is evident that $\tilde{A}$ is a symmetric extension of A, since if

$$h_1, h_2 \in \mathrm{D}_{\tilde{A}} \ \ (h_i = f_i + g_i, \ f_i \in \mathrm{D}_A, \ g_i \in \mathfrak{N}_0, \ i = 1, 2),$$

then

$$\begin{aligned}(\tilde{A}h_1, h_2) &= (Af_1, f_2 + g_2) = (Af_1, f_2) = \\ &= (f_1, Af_2) = (f_1 + g_1, Af_2) = (h_1, \tilde{A}h_2).\end{aligned}$$

Furthermore, it is evident that the subspace $\mathfrak{N}_0$ reduces $\tilde{A}$ since $\mathfrak{N}_0 \subset \mathrm{D}_{\tilde{A}}$ and $\tilde{A}h = 0$ for $h \in \mathfrak{N}_0$. Both $\mathfrak{N}_0$ and its orthogonal complement Δ_A are reduced by the operator $\tilde{A}$. Let $\tilde{A}'$ and $\tilde{A}''$ be the restrictions of $\tilde{A}$ to Δ_A and $\mathfrak{N}_0$.

The range of the operator $\tilde{A}'$ is the subspace Δ_A and, hence, (cf. Section 41), $\tilde{A}'$ is self-adjoint. Since $\tilde{A}'' = 0$, the operator

$$\tilde{A} = \tilde{A}' \oplus \tilde{A}''$$

is also self-adjoint.

Furthermore, for

$$h = f + g \qquad (f \in \mathrm{D}_A, \ g \in \mathfrak{N}_0),$$

we have

$$(\tilde{A}h, h) = (Af, f + g) = (Af, f) \geqq \|f\|^2 \geqq 0,$$

so that the lower bound of the operator $\tilde{A}$ is not negative. Thus, the theorem is proved for the case with $m = 1$ and $m' = 0$. It is easy to see that the general case is reduced to the case considered by the linear transformation

$$A_1 = \frac{1}{m - m'} A - \frac{m'}{m - m'} E.$$

The theorem proved above was established by J. von Neumann. In this connection J. von Neumann made the conjecture that the theorem is true with $m' = m$. Later this conjecture was proved by Stone and Friedrichs, but these authors left open the question of the uniqueness of the extension with preservation of the lower bound.

A complete theory of the self-adjoint extensions of semi-bounded operators with preservation of the lower bound and also the application of this theory to differential equations is due to Krein.[5] A presentation of these results exceeds the scope of this book.

Concluding the present section, we prove a proposition which is a generalization of the theorem of Section 33. There we introduced the notion of inequality between projection operators. Now we introduce the concept of inequality between bounded self-adjoint operators. We write

$$B \geqq A$$

if $B - A$ is a positive operator.

LEMMA: *A monotone increasing sequence of positive self-adjoint operators*

$$0 < A_1 < A_2 < A_3 < \dots$$

with bounded norms

$$\| A_n \| \leqq N \qquad (n = 1, 2, 3, \dots)$$

converges strongly to some limit operator.

Proof: First we remark that if B is a positive self-adjoint operator with norm less than or equal to one, then $B^2 \leqq B$, i.e.,

$$(B^2 f, f) \leqq (Bf, f).$$

This fact follows immediately from the spectral representation of the operator B, but it can be established also without the application of the spectral representation in the following way. Since $\| B \| \leqq 1$,

$$(Bf, f) \leqq (f, f),$$

and, hence,

$$E - B \geqq 0.$$

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[5] cf. M. G. Krein [3], [4], Vol. I.

Also

$$(E - B)\,B\,(E - B) \geqq 0, \quad B\,(E - B)\,B \geqq 0,$$

since

$$(\{E-B\}\,B\,\{E-B\}f,f) = (B\,\{E-B\}f,\,\{E-B\}f) \geqq 0,$$
$$(B\,\{E-B\}\,Bf,f) = (\{E-B\}\,Bf,\,Bf) \geqq 0.$$

But, on the other hand,

$$(E-B)\,B(E-B) + B(E-B)\,B = B - B^2.$$

Hence, we conclude that

$$B - B^2 \geqq 0.$$

Turning now to the proof of the lemma we let $N = 1$ without loss of generality. Since the numerical sequence $\{(A_n f, f)\}_1^\infty$, where f is a fixed element of H, is monotone and bounded, it converges to a finite limit. Therefore, for sufficiently large p_ε and for $n > m > p_\varepsilon$

$$(\{A_n - A_m\}\,f,f) \leqq \epsilon.$$

On the other hand the identity

$$(\{A_n - A_m\}f,f) + (A_m f,f) = (A_n f,f)$$

and the positivity of the summands yield

$$\| A_n - A_m \| \leqq 1.$$

By the remark which was made in the beginning of the proof, we obtain

$$\|(A_n - A_m)f\|^2 = (\{A_n - A_m\}f,\,\{A_n - A_m\}f) =$$
$$= (\{A_n - A_m\}^2 f,f) \leqq (\{A_n - A_m\}f,f) \leqq \epsilon,$$

and the lemma is proved.

86. Some Remarks about the General Theory of Extensions

The presentation in the present chapter of the theory of extensions is based essentially on the assumption that the domain of the operator to be extended is dense in H. This assumption was included in Section 41 for the definition of the concept of a symmetric operator.

Some questions reduce, however, to the consideration of operators with domain not dense in H which satisfy the conditions

$$(Af, g) = (f, Ag) \qquad (f, g \in \mathrm{D}_A).$$

Such operators are also called symmetric. The theory of extensions of von Neumann does not apply to them. In particular, this theory leaves open even the question of the existence of self-adjoint extensions for bounded operators which are symmetric in the extended sense. This last question was resolved in the affirmative by Krein[6], who established the existence of self-adjoint extensions of a bounded operator with preservation of the norm, described these extensions and also gave a criterion for their uniqueness.

Recently Krasnoselski[7] considered the question of symmetric extensions of symmetric operators with nondense domain in complete generality. He showed that every closed symmetric operator has maximal symmetric extensions. If the deficiency indices of such an operator are equal (and only in that case) then it has self-adjoint extensions. The self-adjoint extensions are characterized by formulas similar to formulas (4) and (4′) of Section 80.

In conclusion it is necessary to remark that for the construction of the theory of extensions in such generality one must assume that the operator has a closure. For operators with nondense domains, the existence of closure is not a consequence of symmetry.

The extension of a symmetric operator with a nondense domain for the case when the closure of the domain contains its range was studied first in a remarkable work of M. A. Naimark in connection with the construction of the theory of generalized spectral functions. The presentation of the theory of M. A. Naimark forms the main part of Appendix I.

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[6] cf. M. G. Krein [3], [4], Vol. I.

[7] cf. M. A. Krasnoselski [2], Vol. I.

Appendix I

GENERALIZED EXTENSIONS AND GENERALIZED SPECTRAL FUNCTIONS OF SYMMETRIC OPERATORS

1. Generalized Resolution of the Identity. Naimark's Theorem[1]

A generalized resolution of the identity is defined as a one parameter family of operators F_t which satisfy the following conditions:

(A) Each difference $F_{t_2} - F_{t_1}$, where $t_2 > t_1$, is a bounded positive operator,

(B) $F_{t-0} = F_t$,

(C) $F_{-\infty} = 0, F_\infty = E$.

In contrast with the definition of the ordinary resolution of the identity (cf. Section 61) it is not assumed here that the operators F_t are projections. Furthermore, we do not assume here the earlier orthogonality condition,

$$(1) \qquad F_u F_v = F_s \qquad (s = \min\{u, v\}),$$

since (1) and (A) would imply that F_t is a projection operator.

In view of the lemma of Section 85 on monotone sequences of operators, any family of operators F_t which satisfy condition (A) can be modified or "normalized" so that condition (B) is also satisfied.

The ordinary resolution of the identity is a special case of the generalized one. Sometimes the latter is called simply the resolution of the identity and the ordinary one is called the orthogonal resolution of the identity.

In terms of F_t we define a positive additive operator function

$$F(\Delta) = F_\Delta = F_{t_2} - F_{t_1}$$

of an interval $\Delta = (t_1, t_2)$, where $t_1 < t_2$.

A simple example of a generalized resolution of the identity is given by

$$F_t = \mu_1 E_t^{(1)} + \mu_2 E_t^{(2)},$$

---

[1] Cf. M. A. Naimark [1], [4], Vol. I.

where $E_t^{(1)}$ and $E_t^{(2)}$ are arbitrary orthogonal resolutions of the identity and μ_1 and μ_2 are any positive numbers, such that $\mu_1 + \mu_2 = 1$.

A more instructive example is given by the operator function F_t which is obtained by the following construction. Let E_t be an orthogonal resolution of the identity of the space H, let G be a subspace of H, and let P be the operator of projection of H onto G. We define

$$F_t = PE_t,$$

and consider F_t as restricted to G. It is easy to see that F_t satisfies conditions (A), (B), and (C) and, hence, is a resolution of the identity (not necessarily orthogonal) of the space G.

Naimark proved that every generalized resolution of the identity of the space H can be obtained by the method described above if one imbeds H in a particular space H^+. For the proof of the theorem of M. A. Naimark we need a certain important method for the construction of Hilbert spaces. We first present this method. For this purpose we introduce the concept of a positive-definite function. A scalar function $\Phi(\mathfrak{f}, \mathfrak{g})$ which is defined for each pair $\mathfrak{f}$, $\mathfrak{g}$ of elements of some set $\mathfrak{R}$ is called positive-definite if

$$\Phi(\mathfrak{f}, \mathfrak{g}) = \overline{\Phi(\mathfrak{g}, \mathfrak{f})}$$

for all $\mathfrak{f}$, $\mathfrak{g}$ in $\mathfrak{R}$ and if for arbitrary $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n$ $(n < \infty)$ in $\mathfrak{R}$ the quadratic form

$$\sum_{i,k=1}^{n} \Phi(\mathfrak{f}_i, \mathfrak{f}_k)\, \xi_i \bar{\xi}_k$$

is non-negative.

An example of a positive-definite function of a vector pair in Hilbert space is the scalar product, since

$$(f, g) = (g, f),$$

and

$$\sum_{i,k=1}^{n} (f_i, f_k)\, \xi_i \bar{\xi}_k = \|\sum_{i=1}^{n} \xi_i f_i\|^2 \geqq 0.$$

On the other hand, if a positive-definite function of a pair of elements of an arbitrary set $\mathfrak{R}$ (no algebraic operations are defined in $\mathfrak{R}$) is given, then this set can be made into a Hilbert space. To be precise, $\mathfrak{R}$ can be imbedded in a Hilbert space H^+ such that the scalar product of $\mathfrak{f}$ and $\mathfrak{g}$ in $\mathfrak{R}$ is defined by

$$(\mathfrak{f}, \mathfrak{g}) = \Phi(\mathfrak{f}, \mathfrak{g}).$$

To construct H^+ we complete $\mathfrak{R}$ to a linear space $\hat{\mathfrak{R}}$, in which we introduce formally the finite sums

$$\hat{\mathfrak{f}} = \sum_{i=1}^{n} \xi_i \mathfrak{f}_i$$

for arbitrary $\mathfrak{f}_i \in \mathfrak{R}$ and arbitrary numbers $\xi_i (i = 1, 2, \ldots, n)$. For elements

$$\hat{\mathfrak{f}} = \sum_{i=1}^{n} \xi_i f_i \quad , \quad \hat{\mathfrak{g}} = \sum_{k=1}^{n} \eta_k \mathfrak{g}_k$$

in $\mathfrak{R}$ we define a scalar product by the formula

$$(\hat{\mathfrak{f}}, \hat{\mathfrak{g}}) = \sum_{i,\, k=1}^{n} \Phi(\mathfrak{f}_i, \mathfrak{g}_k)\, \xi_i \bar{\eta}_k.$$

The scalar product thus defined has all the properties listed in Section 2, Chapter 1, with the exception of perhaps one; namely, it is possible that

$$(2) \qquad (\hat{\mathfrak{f}}, \hat{\mathfrak{f}}) = 0 \qquad \text{for } \hat{\mathfrak{f}} \neq 0.$$

If we retrace the proof of the Cauchy-Bunyakovski inequality in Section 2, we see that this proof is valid for the scalar product defined above (strictly speaking, we should call it a *quasiscalar product*). Thus,

$$(3) \qquad |(\hat{\mathfrak{f}}, \hat{\mathfrak{g}})|^2 \leqq (\hat{\mathfrak{f}}, \hat{\mathfrak{f}})(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}).$$

From this inequality it follows that the set $\hat{\mathfrak{N}}$ of all elements $\hat{\mathfrak{f}}$ for which $(\hat{\mathfrak{f}}, \hat{\mathfrak{f}}) = 0$ is a linear manifold.

In order to remove the defect indicated by (2), we introduce the quotient space

$$\mathrm{R}^+ = \hat{\mathfrak{R}}/\hat{\mathfrak{N}}.$$

Elements of the space R^+ are the sets in $\hat{\mathfrak{F}} \subset \hat{\mathfrak{R}}$ such that if $\hat{\mathfrak{f}}_0 \in \hat{\mathfrak{F}}$, then $\hat{\mathfrak{F}}$ consists of all the elements

$$\hat{\mathfrak{f}} = \hat{\mathfrak{f}}_0 + \hat{\mathfrak{g}},$$

where $\hat{\mathfrak{g}}$ is an arbitrary element of $\hat{\mathfrak{N}}$. Multiplication of an element $\hat{\mathfrak{F}}$ of R^+ by a scalar λ is defined such that the product $\lambda\hat{\mathfrak{F}}$ consists of all the elements $\lambda\hat{\mathfrak{f}}$ where $\hat{\mathfrak{f}} \in \hat{\mathfrak{F}}$. The sum in R^+ is defined analogously.

The scalar product of two elements $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{G}}$ in R^+ is defined by the equation

$$(4) \qquad (\hat{\mathfrak{F}}, \hat{\mathfrak{G}}) = (\hat{\mathfrak{f}}, \hat{\mathfrak{g}}) \qquad (\hat{\mathfrak{f}} \in \hat{\mathfrak{F}},\ \hat{\mathfrak{g}} \in \hat{\mathfrak{G}}).$$

Using (3), it is easy to conclude that the scalar product $(\hat{\mathfrak{F}}, \hat{\mathfrak{G}})$ does not depend on the choice of the representatives $\hat{\mathfrak{f}}$ and $\hat{\mathfrak{g}}$ and, hence, that definition (4) is unambiguous. The scalar product $(\hat{\mathfrak{F}}, \hat{\mathfrak{G}})$ now has without exception all the properties listed in Section 2. The completion (cf. Section 3) of the inner product space R^+ is a Hilbert space which we denote by H^+. This space H^+ is the one sought.

Now we come to the theorem of Naimark.

THEOREM: *Let F_t be an arbitrary resolution of the identity for the space* H. *Then there exists a Hilbert space* H^+ *which contains* H *as a subspace and there exists an orthogonal resolution of the identity E_t^+ for the space* H^+ *such that*

$$F_t f = P^+ E_t^+ f$$

for each $f \in \mathrm{H}$ where P^+ is the operator of projection on H.

Proof: Consider the set $\mathfrak{R}$ of all pairs $\mathfrak{p}$ of the form

$$\mathfrak{p} = \{\Delta, f\},$$

where Δ is an arbitrary real interval and f is an arbitrary vector of H. On $\mathfrak{R}$ we define a function $\Phi(\mathfrak{p}_1, \mathfrak{p}_2)$ such that if $\mathfrak{p}_1 = \{\Delta_1, f_1\}$ and $\mathfrak{p}_2 = \{\Delta_2, f_2\}$, then

$$\Phi(\mathfrak{p}_1, \mathfrak{p}_2) = (F_{\Delta_1 \cdot \Delta_2} f_1, f_2).$$

We show that the function $\Phi(\mathfrak{p}_1, \mathfrak{p}_2)$ is positive-definite. Indeed,

$$\Phi(\mathfrak{p}_1, \mathfrak{p}_2) = (F_{\Delta_1 \cdot \Delta_2} f_1, f_2) = (f_1, F_{\Delta_1 \cdot \Delta_2} f_2) = \overline{(F_{\Delta_1 \cdot \Delta_2} f_2, f_1)} = \overline{\Phi(\mathfrak{p}_2, \mathfrak{p}_1)}$$

and, on the other hand,

$$\sum_{i,k=1}^{n} \Phi(\mathfrak{p}_i, \mathfrak{p}_k)\, \xi_i \bar{\xi}_k = \sum_{i,k=1}^{n} (F_{\Delta_i \cdot \Delta_k} f_i, f_k)\, \xi_i \bar{\xi}_k. \tag{5}$$

If the intervals Δ_i $(i = 1, 2, \ldots, n)$ are pairwise disjoint, then

$$\sum_{i,k=1}^{n} (F_{\Delta_i \cdot \Delta_k} f_i, f_k)\, \xi_i \bar{\xi}_k = \sum_{i=1}^{n} (F_{\Delta_i} f_i, f_i)\, |\xi_i|^2 \geqq 0. \tag{6}$$

If the intervals $\Delta_i (i = 1, 2, \ldots, n)$ are pairwise disjoint and the intervals Δ_1 and Δ_2 coincide, then the sums in the right member of (5) fall into two parts. One part, with indices from 3 to n, is of the form (6), and the other part, with indices 1 and 2, satisfies

$$\sum_{i,k=1}^{2} (F_{\Delta_i \cdot \Delta_k} f_i, f_k)\, \xi_i \bar{\xi}_k = \sum_{i,k=1}^{2} (F_{\Delta_1} f_i, f_k)\, \xi_i \bar{\xi}_k = (F_{\Delta_i} \sum_{i=1}^{2} \xi_i f_i, \sum_{k=1}^{2} \xi_k f_k) \geqq 0.$$

The case with arbitrary intervals $\Delta_i (i = 1, 2, \ldots, n)$ can be reduced, with the aid of additional partitions, to the cases already considered. Hence, if $\Delta_1 \cap \Delta_2 = 0$, then

$$(F_{(\Delta_1 + \Delta_2) \cdot \Delta_3} f, g) = (F_{\Delta_1 \cdot \Delta_3 + \Delta_2 \cdot \Delta_3} f, g) = (F_{\Delta_1 \cdot \Delta_3} f, g) + (F_{\Delta_2 \cdot \Delta_3} f, g).$$

Thus, $\Phi(\mathfrak{p}_1, \mathfrak{p}_2)$ is a positive-definite function on $\mathfrak{R}$.

Using the method described earlier we imbed $\mathfrak{R}$ in a Hilbert space H^+.

Not desiring to introduce new notations for those elements $\mathfrak{P}$ of the space H^+ which are subsets of $\mathfrak{R}$ by the construction described earlier, we agree on the following: if an element $\mathfrak{p}$ of $\mathfrak{R}$ belongs to $\mathfrak{P}$ then we write $\mathfrak{p}$ instead of $\mathfrak{P}$.

We indicate the scalar product in the space H^+ by the symbol $_+$, and have

$$(\mathfrak{p}_1, \mathfrak{p}_2)_+ = \Phi(\mathfrak{p}_1, \mathfrak{p}_2).$$

We now consider elements of H^+ of the form $\{I, f\}$, $I = [-\infty, \infty]$. By means of the equation

$$(\{I, f\}, \{I, g\})_+ = (F_I f, g) = (f, g),$$

we can identify the pair $\{I, f\}$ with the element f from H. The element $\sum_{k=1}^{n} \xi_k \{I, f_k\}$ of the space H^+ is identified with the element $\sum_{k=1}^{n} \xi_k f_k$ of the space H. Thus, H can be considered as a subspace of the space H^+.

We now solve the following problem: find the projection of the element $\{\Delta, f\}$ of the space H^+ on the subspace H. We denote the projection to be found by $\{I, g\}$. For each h of H,

$$(\{\Delta, f\} - \{I, g\}, \{I, h\})_+ = 0,$$

or

$$(\{\Delta, f\}, \{I, h\})_+ - (\{I, g\}, \{I, h\})_+ = (F_\Delta f, h) - (g, h) = (F_\Delta f - g, h) = 0,$$

so that

$$g = F_\Delta f,$$

i.e.

$$P^+\{\Delta, f\} = \{I, F_\Delta f\}. \tag{7}$$

The theorem will be proved if it is established that the operator function E_Δ^+, which is defined by

$$E_\Delta^+\{\Delta', f\} = \{\Delta \cap \Delta', f\} \tag{8}$$

for each element of the form $\{\Delta', f\} \in \mathrm{H}^+$ is an orthogonal resolution of the identity for the space H^+, since then (7) can be expressed in the form

$$P^+ E_\Delta^+ f = P^+ E_\Delta^+ \{I, f\} = P^+\{\Delta \cap I, f\} = P^+\{\Delta, f\} = \{I, F_\Delta f\} = F_\Delta f$$

for each $f \in \mathrm{H}$.

It is evident that E_Δ^+ is an additive operator function of an interval. Furthermore, the two equations

$$(E_\Delta^+)^2 \{\Delta', f\} = E_\Delta^+ \{\Delta \cap \Delta', f\} = \{\Delta \cap \Delta \cap \Delta', f\} = E_\Delta^+ \{\Delta', f\},$$

and

$$(E_\Delta^+\{\Delta',f\}, \{\Delta'', g\})_+ = (\{\Delta \cap \Delta',f\}, \{\Delta'',g\})_+ =$$
$$= (F_{\Delta\cdot\Delta'\cdot\Delta''} f,g) = (F_{\Delta'\cdot\Delta\cdot\Delta''} f,g) = (\{\Delta',f\}, E_\Delta^+ \{\Delta'',g\})_+,$$

imply that E_Δ^+ is a projection operator. Finally, it is evident that $E_I^+\{\Delta',f\} = \{\Delta',f\}$.

Since the family of all elements of the form $\{\Delta',f\}$ is dense in H^+, the extension to H^+ by continuity of the operator E_Δ^+ defined by formula (8) is an orthogonal resolution of the identity for the space H^+. The theorem is proved.

2. Self-Adjoint Extensions to Larger Spaces and Spectral Functions of Symmetric Operators[2]

In the present section we apply the concept of the orthogonal sum of the spaces $H_1 \oplus H_2$. A special case ($H_1 \oplus H_1$) of this concept occurred in the definition of the graph of an operator (cf. Section 46). The orthogonal sum

$$H^+ = H_1 \oplus H_2. \tag{1}$$

of two arbitrary Hilbert spaces H_1 and H_2 is the set of all pairs $f^+ = \{f_1, f_2\}$ with $f_1 \in H_1$ and $f_2 \in H_2$. The algebraic operations and the metric are defined analogously to those introduced in Section 46.

It is evident that H^+ is a Hilbert space. If H_1 and H_2 are identified with the sets of all pairs $\{f_1, 0\}$ and $\{0, f_2\}$ with $f_1 \in H_1$ and $f_2 \in H_2$, then H_1 and H_2 can be regarded as mutually orthogonal subspaces of H^+. After the construction of H^+, formula (1) can be considered as a decomposition of H^+ into an orthogonal sum.

A generalization of the concept of a symmetric extension is basic for our further considerations. Let A be a symmetric operator which is defined in a space H and let H^+ be a Hilbert space which contains H. In supplement to the definition in Section 41, we shall call each symmetric (in particular, self-adjoint) operator B^+ which is defined in H^+ and is an extension of the operator A, a symmetric (in particular, self-adjoint) extension of the operator A.

The concept of an isometric (in particular, unitary) extension of an isometric operator is generalized analogously. It is evident that the Cayley transform of a symmetric extension B^+ of an operator A is an isometric

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[2] Cf. M. A. Naimark [2], [1], Vol. I. Before Naimark constructed the general theory which is presented in this and the following section, separate results were obtained by different methods by A. I. Plesner [4], Vol. I, and N. I. Akhiezer [4], Vol. I.

(respectively, unitary) extension of the Cayley transform of the operator A. Conversely, the Cayley transform of an isometric (unitary) extension of an isometric operator is a symmetric (respectively, self-adjoint) extension of the Cayley transform of this operator.

For $\mathrm{H}^+ = \mathrm{H}$ we obtain the ordinary symmetric and isometric extensions which have been considered heretofore.

Let B^+ be an arbitrary symmetric extension of the operator A. Then the following relation holds:

$$\mathrm{D}_A \subset \mathrm{D}_{B^+} \cap \mathrm{H} \subset \mathrm{D}_{B^+}.$$

It is convenient to classify the symmetric extensions B^+ of an operator A by the following scheme.

Extension of type I: $\mathrm{D}_A \neq \mathrm{D}_{B^+} \cap \mathrm{H} = \mathrm{D}_{B^+}$; this extension coincides with the ordinary one.

Extension of type II: $\mathrm{D}_A = \mathrm{D}_{B^+} \cap \mathrm{H} \neq \mathrm{D}_{B^+}$.

Extension of type III. $\mathrm{D}_A \neq \mathrm{D}_{B^+} \cap \mathrm{H} \neq \mathrm{D}_{B^+}$.

Thus, the ordinary symmetric extensions (i.e., those which do not extend beyond the space) are extensions of the first type, and symmetric extensions which do go beyond the space are of types II and III. It is evident that a maximal operator has only symmetric extensions of type II.

If a symmetric extension B^+ of an operator A is reduced by a subspace $\mathrm{G}^+ \subset \mathrm{H}^+ \ominus \mathrm{H}$, then we shall always exclude G^+ from H^+ (i.e., we replace the space H^+ by the space $\mathrm{H}^+ \ominus \mathrm{G}^+$ and the operator B^+ by its restriction to $\mathrm{H}^+ \ominus \mathrm{G}^+$). Under this condition, a self-adjoint operator admits no symmetric extensions.

We now turn to the fundamental theorem of the present section.

THEOREM 1: *Every symmetric operator A defined in a Hilbert space with arbitrary deficiency indices* $(\mathfrak{m}, \mathfrak{n})$ *can be extended to a self-adjoint operator B^+ which is defined in a space* $\mathrm{H}^+ \supset \mathrm{H}$.

Proof: In some space H' of sufficiently large dimension we construct a symmetric operator A' with deficiency indices $(\mathfrak{n}, \mathfrak{m})$. (One can, for example, choose H' isomorphic to H and A' isomorphic to $(-1)\,A$). We now construct the space

$$\mathrm{H}^+ = \mathrm{H} \oplus \mathrm{H}',$$

and we introduce in this space the symmetric operator

$$A^+ = A \oplus A'.$$

It is evident that A^+ is a symmetric extension of A of type II. For the proof of the theorem it is sufficient to establish that the operator A^+ can be extended to a self-adjoint operator. Hence, we must verify that A^+ has equal deficiency indices. For $\Im z \neq 0$,

$$\mathfrak{M}_z^+ = (A^+ - zE^+)\mathrm{D}_{A^+} = (A^+ - zE^+)(\mathrm{D}_A \oplus \mathrm{D}_{A'}) = \\ = (A - zE)\mathrm{D}_A \oplus (A' - zE')\mathrm{D}_{A'} = \mathfrak{M}_z \oplus \mathfrak{M}'_z,$$

or, passing to the orthogonal complement of the corresponding manifolds $\mathfrak{M}_z^+$, $\mathfrak{M}_z$ and $\mathfrak{M}'_z$ in the spaces H^+, H and H$'$,

$$\mathfrak{N}_z^+ = \mathfrak{N}_z \oplus \mathfrak{N}'_z.$$

It follows that the deficiency indices of the operator A^+ are $(\mathfrak{m} + \mathfrak{n}, \mathfrak{m} + \mathfrak{n})$. Thus, the theorem is proved.

If we choose the space H$'$ and the operator A' in a different manner, then we obtain different self-adjoint extensions of A.

The operator B^+ constructed in the proof of Theorem 1 is, generally speaking, an extension of type III of the initial symmetric operator A. However, it is always possible to require that B^+ be an extension of type II of the operator A. For this, it is sufficient to construct a unitary operator U^+ such that

$$U^+\mathfrak{N}_z = \mathfrak{N}'_{\bar{z}},$$

and to define B^+ as the Cayley transform of U^+.

By the theorem just proved, we show now that an arbitrary symmetric operator has an integral representation similar to the one for self-adjoint operators which was derived in Chapter VI. Thus, let A be a symmetric operator in H. We extend[3] it to a self-adjoint operator B^+ and we pass from the space H to a larger space H^+. Let $E^+(\Delta)$ be the spectral function of the operator B^+ and P^+ the operator of projection of H^+ on H. Finally, let

$$F(\Delta) = P^+E^+(\Delta).$$

For arbitrary elements $f \in \mathrm{D}_{B^+}$ and $g \in \mathrm{H}^+$,

$$(B^+f, g) = \int_{-\infty}^{\infty} t\,d(E_t^+ f, g),$$

$$\| B^+f \|^2 = \int_{-\infty}^{\infty} t^2 d(E^+f, f).$$

In particular, if $f \in \mathrm{D}_A$ and $g \in \mathrm{H}$, then these formulas can be written in the form

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[3] We do not assume here that the extension is defined by the special procedure used in the proof of Theorem 1.

$$(2) \qquad (Af, g) = \int_{-\infty}^{\infty} t\, d(F_t f, g),$$

$$(3) \qquad \| Af \|^2 = \int_{-\infty}^{\infty} t^2 d(F_t f, f).$$

Thus, we obtain an integral representation of an arbitrary symmetric operator which is similar to the integral representation of a self-adjoint operator.

In view of similarity of the formulas (2) and (3) to the representation obtained in Section 66 for a self-adjoint operator, we now give the following definition which generalizes the concept of the spectral function.

DEFINITION: *If A is a symmetric operator and F_t is a resolution of the identity such that formulas* (2) *and* (3) *hold for arbitrary $f \in \mathrm{D}_A$ and $g \in \mathrm{H}$, then F_t is a spectral function of the operator A.*

Before we compare formulas (2) and (3) with the integral representation of a self-adjoint operator obtained earlier, we show that the method used in the derivation of these formulas is general. Namely, the following theorem holds.

THEOREM 2: *Every spectral function of a symmetric operator A which is defined in* H *has the form*

$$F_t = P^+ E_t^+,$$

where E_t^+ is the spectral function of some self-adjoint extension B^+ of the operator A, obtained with the aid of an extension of H *to* $\mathrm{H}^+ \supset \mathrm{H}$, *and P^+ is the projection operator of* H^+ *on* H.

Proof: By means of the theorem of Naimark we construct the space H^+ and, in it, an orthogonal resolution of the identity E_t^+ such that

$$(4) \qquad F_t = P^+ E_t^+.$$

We show that the operator B^+, defined by

$$\int_{-\infty}^{\infty} t d E_t^+ f$$

for all $f \in \mathrm{H}^+$ which satisfy the inequality

$$\int_{-\infty}^{\infty} t^2 d(E_t^+ f, f) < \infty,$$

is a self-adjoint extension of the operator A. From Theorem 1 of Section 66 it follows that the operator B^+ is self-adjoint.

If $f \in D_A$, then $f \in D_{B^+}$, since

$$\int_{-\infty}^{\infty} t^2 d(E_t^+ f, f) = \int_{-\infty}^{\infty} t^2 d(F_t f, f) = \| Af \|^2 < \infty.$$

Furthermore, for $f \in D_A$ and $g \in H$, it follows from the equation

$$(Af, g) = \int_{-\infty}^{\infty} t d(F_t f, g) = \int_{-\infty}^{\infty} t d(E_t^+ f, g) = (B^+ f, g)$$

that

$$Af = P^+ B^+ f \qquad (f \in D_A). \tag{5}$$

But, for $f \in D_A$,

$$\| Af \|^2 = \int_{-\infty}^{\infty} t^2 d(F_t f, f) = \int_{-\infty}^{\infty} t^2 d(E_t^+ f, f) = \| B^+ f \|^2. \tag{6}$$

From (5) and (6) it follows that

$$Af = B^+ f \qquad (f \in D_A).$$

It can be shown that the operator B^+ is reduced by no subspace of $H^+ \ominus H$. Indeed, if a subspace G of $H^+ \ominus H$ reduces the operator B^+ then it also reduces the resolution of the identity E_t^+ and the exclusion of G from H^+ will reduce to the exclusion of the restriction of the operator E_t^+ to G. This does not affect formula (4). Thus, the theorem is proved.

If a spectral function F_t of a symmetric operator is represented in the form (4), where E_t^+ is the spectral function of a self-adjoint extension B^+ of the operator A, then the spectral function F_t is *generated by* the self-adjoint extension B^+. Thus, every self-adjoint extension of the operator A generates some spectral function of this operator, and conversely, every spectral function of the operator A is generated by one of its self-adjoint extensions.

We present now several facts which will clarify the interrelation between the new and old definitions of a spectral function and which indicate to what extent there is an analogy between the representation (2), (3) and the spectral representation of a self-adjoint operator.

1° The spectral function of a self-adjoint operator defined in Chapter VI is its unique spectral function in the sense of the definition of the present section. A self-adjoint operator does not have other spectral functions since, by our theorem, every such spectral function must be generated by a self-adjoint extension and a self-adjoint operator does not have such extensions.

2° From formula (3) it follows that the integral

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) \tag{7}$$

converges for each $f \in \mathrm{D}_A$. The converse statement, generally speaking, is not correct. It is possible to assert only that if the integral (7) converges for some $f \in \mathrm{H}$ then

$$\int_{-\infty}^{\infty} t^2 d(E_t^+ f, f) < \infty. \tag{8}$$

Thus the vector f belongs to the manifold $\mathrm{D}_{B+} \cap \mathrm{H}$. This coincides with D_A if and only if B^+ is an extension of type II of the operator A.

Thus, only those spectral functions of the operator A which are generated by self-adjoint extensions of type II of A characterize the domain D_A as the set of vectors for which the integral (7) converges. In particular, the domains of maximal operators are characterized by the spectral functions and by inequality (8), since such operators admit only extensions of type II.

In the general case the spectral function F_t of the operator A, generated by a self-adjoint extension B^+, characterizes the domain of the operator $C = P^+B^+P^+$. This is a symmetric extension of type I of A with a maximal domain D_C which satisfies the condition

$$\mathrm{D}_A \subset \mathrm{D}_C \subset \mathrm{D}_{B+}. \tag{9}$$

With this point of view it would be natural to consider the resolution of the identity F_t not as a spectral function of A but of C and, in correspondence with this, to include in the definition the requirement that the resolution of the identity F_t, apart from the representations (2), (3), define still the domain D_A by the inequality (8). However, it seems impractical to include in the definition an additional requirement concerning the characterization of the domain D_A. Thus, the definition admits that a single resolution of the identity may be the spectral function of two different operators (for example, the operators A and $C = P^+B^+P^+$). In particular, the spectral function of a self-adjoint extension of type I of a given operator A with equal deficiency numbers is the spectral function of the same operator A and of each operator C such that $C \subset B$.

3° From the preceding it follows that every symmetric operator corresponds to some set of spectral functions and that one and the same spectral function corresponds to some set of symmetric operators.

In connection with this, there arises the question: is every resolution of the identity the spectral function of a symmetric operator? In the case of the orthogonal resolution of the identity a positive answer to this question was given by Theorem 1, Section 66.

If, however, F_t is a nonorthogonal resolution of the identity then it is impossible to assert that the set of vectors for which the integral on the right side of equation (3) exists will be dense in the space H. In fact, as Naimark observed, there exist resolutions of the identity such that the integral on the right side of equation (3) does not converge for any vector $f \neq 0$. As such an example,[4] we consider the operator function defined on the space H by the equations

$$F_t = \begin{cases} 0 & t \leqq 0 \\ e^{-\frac{1}{t}} E & t > 0. \end{cases}$$

It is evident, the operator function F_t satisfies conditions A, B, C in section 1° and, hence, is a resolution of the identity. However this resolution of the identity is not a spectral function of any symmetric operator, since the integral

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) = (f, f) \int_{0}^{\infty} t^2 d e^{-\frac{1}{t}} = (f, f) \int_{0}^{\infty} e^{-\frac{1}{t}} dt$$

diverges for each vector $f \neq 0$.

4° In Chapter VI it was shown that for a self-adjoint operator the representation (2) holds in the "strong" sense. In the case of the nonorthogonal resolution of the identity the transition from the "weak" representation (2) to the "strong" representation

$$Af = \int_{-\infty}^{\infty} t dF_t f \tag{10}$$

is impossible. Therefore, we regard equation (10) only as a symbolic entry of the formulas (2) and (3). The concept of the integral in the strong sense in the case of nonorthogonal resolutions of the identity is not introduced.

5° From the property of the orthogonality of the spectral function E_t of a self-adjoint operator, it follows that for each finite real interval Δ, the vector $E(\Delta)f$ with $f \in$ H belongs to D_A and

$$(AE(\Delta)f, g) = \int_{\Delta} t \, d(E_t f, g)$$

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[4] For this example we thank M. A. Krasnoselski.

for each $g \in \mathrm{H}$. This important fact which we applied repeatedly in Chapter VI does not hold for the integral representation of symmetric operator considered here (even if we restricted ourselves to spectral functions generated by extensions of type II).

However the following proposition is correct: if A is a symmetric operator, if F_t is its spectral function, if Δ is a finite real interval, and if g is an arbitrary vector in H, then for each $h \in \mathrm{H}$,

$$F(\Delta)\, g \in \mathrm{D}_A{}^*$$

and

$$(A^*F(\Delta)\, g, h) = \int_\Delta t\, d\,(F_t g, h).$$

Indeed, if B^+ is the extension which generates F_t and if P^+ is the operator of projection of H^+ on H, then for each $f \in \mathrm{D}_A$

$$(11)\quad (Af, F(\Delta)g) = (B^+f, P^+E^+(\Delta)g) = (B^+f, E^+(\Delta)g) = \\ = \int_\Delta t\, d(E_t^+f, g) = \int_\Delta t\, d(E_t^+f, P^+g) = \int_\Delta t\, d(F_tf, g).$$

But the integral

$$\int_\Delta t\, d\,(F_t h, g)$$

is a bilinear functional of h and g on H and, therefore, has a representation (h, Cg) where C is a bounded operator. It follows that

$$(Af, F(\Delta)\, g) = (f, g^*),$$

and, hence,

$$F(\Delta)\, g \in \mathrm{D}_{A^*}.$$

Therefore, (11) yields

$$(f, A^*F(\Delta)\, g) = \int_\Delta t\, d\,(F_tf, g).$$

Since D_A is dense in H, this equation holds for an arbitrary vector $h \in \mathrm{H}$.

3. Spectral Functions of Symmetric Operators and Generalized Resolvents

We now consider an operator function $\mathbf{R}_z$, which is related to the spectral function of a symmetric operator in the same way that the resolvent of a self-adjoint operator is related to its spectral function.

DEFINITION: *Let A be a symmetric (but not self-adjoint) operator. Let B^+ be a self-adjoint extension of A such that B^+ is defined in a space* $\mathrm{H}^+ \supset \mathrm{H}$. *Let R_z^+ be the resolvent of B^+. Finally, let P^+ be the operator of projection of* H^+ *on* H. *The operator* $\mathbf{R}_z$ *defined on* H *by*

$$\mathbf{R}_z = P^+ R_z^+$$

for each nonreal z, is called a generalized resolvent of the operator A.

If the operator B^+ and $\mathbf{R}_z$ are related as indicated in the definition, then we say that the generalized resolvent $\mathbf{R}_z$ is generated by the self-adjoint extension B^+.

The generalized resolvents of an operator A (with equal deficiency numbers) generated by its self-adjoint extensions of type I are called orthogonal; these generalized resolvents are also the ordinary resolvents of the self-adjoint extensions which generate them.

The following theorem establishes an integral representation for a generalized resolvent, analogous to the one obtained in Chapter VI for an ordinary resolvent of a self-adjoint operator.

THEOREM 1: *In order that the operator function* $\mathbf{R}_z(\Im z \neq 0)$ *be a generalized resolvent of the symmetric operator A, it is necessary and sufficient that it have the representation*

$$(1) \qquad (\mathbf{R}_z f, g) = \int_{-\infty}^{\infty} \frac{d(F_t f, g)}{t - z} \qquad (f, g \in \mathrm{H}),$$

where F_t is some spectral function of the operator A.

Proof: For the necessity of the representation (1) we may take the spectral function F_t corresponding to that self-adjoint extension B^+ which generates the generalized resolvent $\mathbf{R}_z$. For the sufficiency we may take the generalized resolvent $\mathbf{R}_z$ generated by that self-adjoint extension B^+ corresponding to the spectral function F_t.

If the generalized resolvent $\mathbf{R}_z$ and the spectral function F_t are related by formula (1), then it is said that they correspond to each other. The well known inversion formula between the set of the spectral functions of a given symmetric operator and the set of its generalized resolvents determines a one-to-one correspondence.

We remark that for every vector of the subspace

$$\mathfrak{M}_z = \Delta_A(z) \qquad (\Im z \neq 0),$$

the values of all generalized resolvents of the operator A coincide. Indeed, if

$$(2) \qquad (A - zE)f = g,$$

then

$$(3) \quad \mathbf{R}_z g = \mathbf{R}_z(A - zE)f = P^+ R_z^+ (B^+ - zE^+)f = P^+ f = f.$$

If the generalized resolvent $\mathbf{R}_z$ is generated by an extension B^+ with the resolvent R_z^+, then for $f, g \in \mathrm{H}$

$$(\mathbf{R}_z f, g) = (P^+ R_z^+ f, g) = (R_z^+ f, g) = (f, R_{\bar{z}}^+ g) = (f, P^+ R_{\bar{z}}^+ g) = (f, \mathbf{R}_{\bar{z}} g),$$

and this yields

$$(4) \qquad \mathbf{R}_{\bar{z}} = \mathbf{R}_z^*.$$

If A is a maximal symmetric operator and

$$\mathfrak{M}_z = \mathrm{H}$$

for $\Im z > 0$, then its generalized resolvents coincide for $\Im z > 0$, since by (2) and (3),

$$\mathbf{R}_z g = (A - zE)^{-1} g$$

for $\Im z > 0$ and $g \in \mathrm{H}$. According to (4), the coincidence of the operators $\mathbf{R}_z$ for $\Im z > 0$ implies that they coincide for $\Im z < 0$. Thus, a maximal operator has a unique generalized resolvent and hence possesses a unique spectral function.

On the other hand, a generalized resolvent of a non-maximal symmetric operator A is not uniquely defined. In order to ascertain this, it is sufficient to take two different maximal symmetric extensions C' and C'' of A of type I and to extend them to self-adjoint operators B'^+ and B''^+. It is evident that the generalized resolvents $\mathbf{R}'_z$ and $\mathbf{R}''_z$ of the operator A generated by the extensions B'^+ and B''^+ do not coincide.

Since a non-maximal symmetric operator has different generalized resolvents, its spectral function is not defined uniquely.

In view of an earlier fact (Section 88, 1°) about spectral functions of self-adjoint operator, the following proposition is valid.

THEOREM 2: *A symmetric operator has a unique spectral function if and only if it is maximal. This unique spectral function is orthogonal if and only if the operator is self-adjoint.*

It is easy to see that the set of all spectral functions (respectively, generalized resolvents) of a symmetric operator A is convex. This means that if F'_t and F''_t ($\mathbf{R}'_z$ and $\mathbf{R}''_z$) are two spectral functions (respectively, generalized resolvents) of the operator A, then for $\alpha' + \alpha'' = 1$, $\alpha' > 0$, $\alpha'' > 0$ the operator function $\alpha' F'_t + \alpha'' F''_t$ (respectively, $\alpha' \mathbf{R}'_z + \alpha'' \mathbf{R}''_z$) is also a spectral function (respectively, generalized resolvent) of A.

In connection with this we indicate one method of construction of generalized resolvents (spectral functions) which are generated by self-adjoint extensions of type II by means of operations inside the space H. Let A be a symmetric operator defined in H, and let A' and A'' be any two of its maximal extensions of type I. For definiteness we assume that

$$\Delta_{A'}(z) = \Delta_{A''}(z) = \mathrm{H}$$

for $\Im z > 0$.

We define the operators $\mathbf{R}'_z$ and $\mathbf{R}''_z$ by the equations

$$R'_z = \begin{cases} (A' - zE)^{-1} & (\Im z > 0), \\ (\mathbf{R}'_{\bar z})^* & (\Im z < 0), \end{cases}$$

$$R''_z = \begin{cases} (A'' - zE)^{-1} & (\Im z > 0), \\ (\mathbf{R}''_{\bar z})^* & (\Im z < 0). \end{cases}$$

The operators $\mathbf{R}'_z$ and $\mathbf{R}''_z$ are the generalized resolvents of the maximal symmetric operators A' and A'' respectively. By means of $\mathbf{R}'_z$ and $\mathbf{R}''_z$ we determine the spectral functions F'_t and F''_t of A' and A'', respectively, and form the operator function

$$F_t = \alpha' F'_t + \alpha'' F''_t \qquad (\alpha' + \alpha'' = 1, \alpha' > 0, \alpha'' > 0).$$

Since the set of the spectral functions is convex, F_t is a spectral function of A.

We show now how to choose the maximal extensions A' and A'' in order that the spectral function F_t is generated by an extension of type II or, equivalently, in order that the spectral function F_t and the inequality

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) < \infty \tag{5}$$

determine the domain D_A. It is easy to see that a vector $f \in \mathrm{H}$ satisfies the condition (5) if and only if

$$f \in \mathrm{D}_{A'} \cap \mathrm{D}_{A''}.$$

It remains to prove that the maximal extensions A' and A'' of the operator A can be chosen such that the intersection of the domains $\mathrm{D}_{A'}$ and $\mathrm{D}_{A''}$ is the domain D_A. By the second Neumann formula (Section 80)

$$\mathrm{D}_{A'} = \mathrm{D}_A + \Gamma',$$
$$\mathrm{D}_{A''} = \mathrm{D}_A + \Gamma'',$$

where Γ' is the set of all vectors of the form

$$g + U'g \qquad (g \in \mathfrak{N}_{\bar z}),$$

and U' is an isometric operator which maps $\mathfrak{N}_{\bar z}$ into $U'\mathfrak{N}_{\bar z} \subset \mathfrak{N}_z (\Im z > 0$, if it is assumed for definiteness that $\mathfrak{m} \leq \mathfrak{n})$; Γ'' is defined analogously.

Note that $D_{A'} \cap D_{A''} \supset D_A$. Let us assume that D_A is a proper subset of $D_{A'} \cap D_{A''}$ and choose a vector $h \notin D_A$ which belongs to both $D_{A'}$ and $D_{A''}$. Then

$$h = f' + g' + U'g',$$
$$h = f'' + g'' + U''g'',$$

where $f', f'' \in D_A$ and $g', g'' \in \mathfrak{N}_{\bar{z}}$. These representations of h yield

$$(f'' - f') + (g'' - g') + (U''g'' - U'g') = 0.$$

The summands in the left member of the equation belong to the manifolds D_A, $\mathfrak{N}_{\bar{z}}$ and $\mathfrak{N}_z$ respectively. Since these manifolds are linearly independent,

$$f' = f'', \quad g' = g'' = g, \tag{6}$$
$$U'g = U''g.$$

From the last equation it follows that if

$$D_{A'} \cap D_{A''} = D_A,$$

then the isometric operators U' and U'' must be defined such that equation (6) holds for no vector of $\mathfrak{N}_{\bar{z}}$ which is different from zero. This condition can always be achieved, for example, by choosing U' arbitrarily and letting $U'' = -U'$.

Thus, the constructed operator functions

$$F_t = \alpha' F_t' + \alpha'' F_t''$$

and

$$\mathbf{R}_z = \alpha' \mathbf{R}_z' + \alpha'' \mathbf{R}_z''$$

for arbitrary α', $\alpha''(\alpha' + \alpha'' = 1,\ \alpha' > 0,\ \alpha'' > 0)$ are spectral functions and generalized resolvents, respectively, of the operator A, which are generated in the sense of Section 2 by a self-adjoint extension B^+ of type II of A.

However, it is not true that such a method yields all spectral functions of a symmetric operator.[5]

From the given method of construction of spectral functions it follows, among other things, that Theorem 2 remains correct even if the spectral functions are required to be generated by self-adjoint extensions of type II of the operator A. In order to prove this it is necessary to show that a non-maximal symmetric operator A has different spectral functions which are generated by self-adjoint extensions of type II of A. But this is a consequence of the fact that the generalized resolvents

$$\alpha \mathbf{R}_z' + \beta \mathbf{R}_z''$$

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[5] In this connection, see M. A. Naimark [5], Vol. I, and I. M. Glazman [2], Vol. I.

and

$$\beta\mathbf{R}'_z + \alpha\mathbf{R}''_z$$

for the indicated selections of the maximal extensions A' and A'' coincide only for vectors in $\Delta_A(z)$ and, therefore, generate different spectral functions.

In conclusion, in this and the following section, we illustrate facts and methods with a differential operator. Let P_0 be the operator of differentiation which acts in $L^2(0, \infty)$. The deficiency indices of the operator P_0 are $(0, 1)$.

To obtain a generalized self-adjoint extension of the operator P_0 by the method of Theorem 1, we introduce the operator P' of differentiation in $L^2(-\infty, 0)$ which is defined by the formula

$$P'_0 = i\frac{d}{dt}$$

for each absolutely continuous function $\varphi(t)$ such that $\varphi(t)$ and its derivative $\varphi'(t)$ belong to $L^2(-\infty, 0)$ and such that $\varphi(t)$ satisfies the boundary condition $\varphi(0) = 0$. It is evident that the deficiency indices of the operator P'_0 are $(1, 0)$.

We form the orthogonal sums

$$L^2(-\infty, \infty) = L^2(-\infty, 0) \oplus L^2(0, \infty),$$
$$P_0^+ = P'_0 \oplus P_0.$$

Obviously, the operator P_0^+ is defined by the formula

$$P_0^+\varphi = i\frac{d}{dt}\varphi(t)$$

for each function $\varphi(t)$ which is absolutely continuous in the intervals $(-\infty, 0)$ and $(0, \infty)$ such that $\varphi(t)$ and $\varphi'(t)$ belong to $L^2(-\infty, \infty)$ and such that $\varphi(t)$ satisfies the boundary condition $\varphi(0) = 0$.

It is easy to see that one obtains the domain $\mathrm{D}_{(P_0^+)^*}$ of the adjoint operator $(P_0^+)^*$ if one omits the boundary condition $\varphi(0) = 0$. Therefore, each of the equations

$$(P_0^+)^*g + ig = 0,$$
$$(P_0^+)^*g - ig = 0$$

has a unique solution. These solutions are defined, respectively, by the formulas

$$g_1(t) = \begin{cases} 0 & (t < 0) \\ e^{-t} & (t \geqq 0) \end{cases} \qquad g_2(t) = \begin{cases} e^{t} & (t < 0) \\ 0 & (t \geqq 0). \end{cases}$$

Thus, the deficiency indices of the operator P_0^+ are (1, 1).

The operator P of differentiation on the entire axis $(-\infty, \infty)$ considered in Section 49 is evidently a self-adjoint extension of the operator P_0^+. Thus, the operator of differentiation on the entire axis is a generalized self-adjoint extension of the operator of differentiation on the semi-axis.

4. The Formula of Krein for Generalized Resolvents

In connection with the results of the preceding section, there is the problem of describing the set of all spectral functions of a given symmetric operator. Since the set of all spectral functions and the set of all resolvents of a symmetric operator have a one-to-one correspondence determined by the formula

$$(\mathbf{R}_\lambda f, g) = \int_{-\infty}^{\infty} \frac{d(F_t f, g)}{t-\lambda},$$

this problem is equivalent to the problem of describing all resolvents. The latter problem was solved by Krein[6] for the case of equal and finite deficiency numbers. We present here the result of Krein for the case of deficiency indices (1, 1).

Let A be a symmetric operator with deficiency indices (1, 1), let $\mathring{A}$ be a fixed self-adjoint extension of type I of A, let $\mathring{R}_\lambda$ be the resolvent of the operator $\mathring{A}$, and finally let $\mathbf{R}_\lambda$ be an arbitrary generalized resolvent of the operator A such that

$$\mathbf{R}_\lambda = P^+ R_\lambda^+.$$

Here R_λ^+ is the orthogonal resolvent of some self-adjoint extension A^+ of the operator A which is defined in a space $\mathrm{H}^+ \supset \mathrm{H}$ and P^+ is the operator of projection of H^+ on H.

We let as usual

$$\mathfrak{M}_\lambda = \Delta_A(\lambda) \qquad (\Im\lambda \neq 0),$$
$$\mathfrak{N}_\lambda \neq \mathrm{H} \ominus \mathfrak{M}_\lambda.$$

In the case considered the dimension of the subspace $\mathfrak{N}_\lambda$ is one.

For the difference of the resolvents we have, as in Section 84,

$$(1) \qquad \begin{cases} (\mathring{R}_\lambda - \mathbf{R}_\lambda) f = 0 & \text{for} \quad f \in \mathfrak{M}_\lambda, \\ (\mathring{R}_\lambda - \mathbf{R}_\lambda) f \in \mathfrak{N}_\lambda & \text{for} \quad f \in \mathfrak{N}_\lambda. \end{cases}$$

---

[6] Cf. M. G. Krein [1], Vol. I. A formula for generalized resolvents of an operator with deficiency indices (1, 1) was obtained by Naimark [1], Vol. I. A further generalization of the formulas of Naimark and Krein was given by Strauss [1], Vol. I.

The last relation follows from the fact that for $f \in \mathfrak{N}_\lambda$ and $h \in \mathfrak{M}_{\bar\lambda}$

$$((\mathring{R}_\lambda - \mathbf{R}_\lambda)f, h) = (\mathring{R}_\lambda f, h) - (R_\lambda^+ f, h) =$$
$$= (f, \mathring{R}_{\bar\lambda} h) - (f, R_{\bar\lambda}^+ h) = (f, (\mathring{R}_{\bar\lambda} - \mathbf{R}_{\bar\lambda})h) = (f, 0) = 0.$$

From (1), repeating the argument of Section 84, we get

$$\mathbf{R}_\lambda f = \mathring{R}_\lambda f - \frac{(f, g(\bar\lambda))\, g(\lambda)}{Q(\lambda)}, \tag{2}$$

where

$$g(\lambda) = g(\lambda_0) + (\lambda - \lambda_0)\mathring{R}_\lambda g(\lambda_0). \tag{2'}$$

Here $g(\lambda_0)$ is a vector in $\mathfrak{N}_{\bar\lambda_0}$ with norm one. We assume for definiteness that the fixed point λ_0 lies in the upper half-plane (it is possible to choose $\lambda_0 = i$). However, generally speaking, the function $Q(\lambda)$ does not satisfy the relation

$$Q(\lambda) = Q(\lambda_0) + (\lambda - \lambda_0)(g(\lambda_0), g(\bar\lambda)), \tag{3}$$

which we obtained in Section 84 for the case of orthogonal resolvents.

Let us turn to the clarification of the nature of the function $Q(\lambda)$. First, we find $Q(\lambda_0)$, for which we let $\lambda = \lambda_0$ in (2) and pass from the resolvent to the Cayley transform

$$\mathring{U}_{\lambda_0} = \frac{\mathring{A} - \bar\lambda_0 E}{\mathring{A} - \lambda_0 E}, \qquad U^+_{\lambda_0} = \frac{A^+ - \bar\lambda_0 E^+}{A^+ - \lambda_0 E^+}.$$

After an elementary calculation, we get

$$P^+ U^+_{\lambda_0} f = \mathring{U}_{\lambda_0} f - \frac{\lambda_0 - \bar\lambda_0}{Q(\lambda_0)}(f, g(\bar\lambda_0))\, g(\lambda_0).$$

In this formula, we let $f = g(\bar\lambda_0)$. Then we have

$$P^+ U^+_{\lambda_0} g(\bar\lambda_0) = g(\lambda_0) - \frac{\lambda_0 - \bar\lambda_0}{Q(\lambda_0)} g(\lambda_0). \tag{4}$$

Now, since the element $\psi = U^+_{\lambda_0}\varphi$ belongs to $\mathfrak{M}_{\bar\lambda_0}$ if and only if φ belongs to $\mathfrak{M}_{\lambda_0}$,

$$0 = (g(\bar\lambda_0), \varphi) = (U^+_{\lambda_0} g(\bar\lambda_0), U^+_{\lambda_0}\varphi) = (U^+_{\lambda_0} g(\bar\lambda_0), \psi) = (P^+ U^+_{\lambda_0} g(\bar\lambda_0), \psi);$$

which implies that

$$P^+ U^+_{\lambda_0} g(\bar\lambda_0) = \theta g(\lambda_0). \tag{5}$$

In (5), θ is a constant which does not exceed one in absolute value, and which is determined by the extension A^+; and θ equals one in absolute value if and only if A^+ is an extension of type I.

Our considerations yield an important consequence: if

$$\mathbf{R}_\lambda = \overset{\circ}{R}_\lambda$$

for any one nonreal value $\lambda = \lambda_1$, *then the equation holds for all nonreal* λ. Indeed, replacing λ_0 in formulas (2) and (2′) by λ_1 ($\Im\lambda_1 > 0$) we find from formula (4) that $Q(\lambda_1) = \infty$, which implies that the parameter θ in formula (5) equals one. Therefore, $\mathbf{R}_\lambda$ is an orthogonal resolvent; and two orthogonal resolvents of one and the same operator which coincide at one point are obviously identical.

Comparing (4) and (5), we find that

$$Q(\lambda_0) = \frac{\lambda_0 - \bar{\lambda}_0}{1-\theta},$$

or

$$Q(\lambda_0) = i\Im\lambda_0 + \tau. \tag{6}$$

Here τ is a new parameter which is related to θ by the formula

$$\tau = i(\Im\lambda_0)\frac{1+\theta}{1-\theta},$$

which maps the unit circle of the θ-plane into the upper half τ-plane.

If $\mathbf{R}_\lambda$ runs through the set of orthogonal resolvents, then, since (3) is valid in this case, formula (2) takes the form

$$\mathbf{R}_\lambda f = \overset{\circ}{R}_\lambda f - \frac{(f, g(\bar{\lambda}))\, g(\lambda)}{\tau + Q_1(\lambda)}, \tag{7}$$

where

$$Q_1(\lambda) = i\Im\lambda_0 + (\lambda - \lambda_0)(g(\lambda_0), g(\bar{\lambda})). \tag{7′}$$

The parameter τ is real ($-\infty < \tau \leqq \infty$). Formula (7) defines a one-to-one correspondence between the set of all orthogonal resolvents and the values of the parameter τ.

In the general case where $\mathbf{R}_\lambda$ is an arbitrary resolvent, we let in formula (2)

$$Q(\lambda) = \tau(\lambda) + Q_1(\lambda).$$

Then

$$\mathbf{R}_\lambda f = \overset{\circ}{R}_\lambda f - \frac{(f, g(\bar{\lambda}))\, g(\lambda)}{\tau(\lambda) + Q_1(\lambda)}. \tag{8}$$

We show that the function $\tau(\lambda)$ is holomorphic and has a non-negative imaginary part in the upper half-plane. With this aim, we remark first that the equation

$$\mathbf{R}_\lambda g(\bar{\lambda}) = \overset{\circ}{R}_\lambda g(\bar{\lambda}),$$

and also the equation

$$(\mathbf{R}_\lambda g(\bar\lambda), g(\lambda)) = (\mathring{\mathbf{R}}_\lambda g(\bar\lambda), g(\lambda))$$

hold at no nonreal point. Therefore, we let $f = g(\bar\lambda)$ in (8) and form the inner product of both sides with $g(\lambda)$. We obtain

$$\frac{1}{\tau(\lambda) + Q_1(\lambda)} = ((\mathring{R}_\lambda - \mathbf{R}_\lambda) g(\bar\lambda), g(\lambda)).$$

But, since $Q_1(\lambda)$ and the right member are holomorphic and the latter is different from zero, the function $\tau(\lambda)$ is holomorphic for all nonreal values λ. We prove next that if $\Im\, \tau(\lambda') = 0$ at any one point of the upper half-plane then $\tau(\lambda) =$ constant. It follows, in particular, that the inequality $\Im\, \tau(\lambda) < 0$ is impossible at any point of the upper half-plane. Thus, let $\Im \tau(\lambda') = 0$ $(\Im \lambda' > 0)$. We choose $\lambda = \lambda'$ in both formulas (7) and (8) and take in formula (7) the number $\tau(\lambda')$ as the constant τ. Then the right members of these formulas will be identical for each f. Hence, at the point $\lambda = \lambda'$ the generalized resolvent $\mathbf{R}_\lambda$ coincides with an orthogonal resolvent R_λ and as we showed above, these resolvents must coincide.

We form the inner product of both sides of equation (8) with f and obtain the formula

$$(\mathbf{R}_\lambda f, f) = \frac{p_0(\lambda) + p_1(\lambda)\,\tau(\lambda)}{q_0(\lambda) + q_1(\lambda)\,\tau(\lambda)}\,.$$

For fixed f and λ the functions $p_0(\lambda)$, $p_1(\lambda)$, $q_0(\lambda)$, $q_1(\lambda)$ will have completely determined values, not depending on the choice of the resolvent $\mathbf{R}_\lambda$, which are determined by the function $\tau(\lambda)$. Since orthogonal resolvents are obtained when $\tau(\lambda)$ is a real constant and also when $\tau(\lambda)$ runs through a particular set of holomorphic functions with non-negative imaginary part, the point $w = (\mathbf{R}_\lambda f, f)$ belongs to some circular region. The boundary of this region, the circle $C(f; \lambda)$ is generated by the point w, when $\mathbf{R}_\lambda$ is an orthogonal resolvent. In other words, the circle $C(f; \lambda)$ is described by the point

$$w = \frac{p_0(\lambda) + p_1(\lambda)\,\tau}{q_0(\lambda) + q_1(\lambda)\,\tau},$$

when τ runs through the real axis.

Since the set of all resolvents of a given operator is convex, the point $w = (\mathbf{R}_\lambda f, f)$ runs through the region $K(f; \lambda)$ with boundary $C(f; \lambda)$, when $\mathbf{R}_\lambda$ runs through the set of all resolvents of the operator A.

By our considerations we have now established that: *every resolvent*

$\mathbf{R}_\lambda$ *of a symmetric operator* A *with the deficiency indices* $(1, 1)$ *is representable in the form*

$$\text{(9)} \qquad \mathbf{R}_\lambda = \overset{\circ}{R}_\lambda - \frac{(\cdot\,, g(\bar{\lambda}))\, g(\lambda)}{\tau(\lambda) + Q_1(\lambda)}\,,$$

where $\tau(\lambda)$ *belongs to the class* N *of all functions which are defined in the half-plane* $\mathfrak{I}\,\lambda > 0$ *and are holomorphic and in this half-plane have a nonnegative imaginary part.*

We now prove the converse proposition: *each function* $\tau(\lambda)$ *of the class* N *generates by means of formula* (9) *a resolvent of the operator* A. Let $\tau(\lambda) \in N$ and let S_λ be an operator defined by the equation

$$\text{(9}'\text{)} \qquad S_\lambda = \overset{\circ}{R}_\lambda - \frac{(\cdot\,, g(\bar{\lambda}))\, g(\lambda)}{\tau(\lambda) + Q_1(\lambda)}\,.$$

Since the circle $C(f; \lambda)$ lies in the upper half-plane, but the point $(S_\lambda f, f)$ lies in the circular region $K(f; \lambda)$, the scalar product $(S_\lambda f, f)$ is a holomorphic function in the half-plane $\mathfrak{I}\,\lambda > 0$ which evidently belongs to the class N.

Furthermore, since, for orthogonal resolvents R_λ, the scalar product $(R_\lambda f, f)$ satisfies the inequality

$$|\,(R_\lambda f, f)\,| \leqq \frac{(f, f)}{\mathfrak{I}\lambda} \qquad (\mathfrak{I}\lambda > 0),$$

the scalar product $(S_\lambda f, f)$, which lies in the circular region $K(f; \lambda)$, satisfies the inequality

$$|\,(S_\lambda f, f)\,| \leqq \frac{(f, f)}{\mathfrak{I}\lambda}\,.$$

Using this inequality and repeating the argument of Section 65, we obtain a representation of the scalar product $(S_\lambda f, f)$ in the form

$$\text{(10)} \qquad (S_\lambda f, f) = \int_{-\infty}^{\infty} \frac{d\omega(t; f)}{t - \lambda}\,.$$

Here $\omega(t; \lambda) = \omega(t)$ is a nondecreasing left-continuous function which tends to zero as $t \to -\infty$ and satisfies the condition

$$\omega(t; f) \leqq (f, f) \qquad (-\infty < t \leqq \infty).$$

From formula (10), repeating the calculation of Section 65, we obtain a representation of the operator S_λ in the form

$$(S_\lambda f, g) = \int_{-\infty}^{\infty} \frac{d(F_t f, g)}{t - \lambda}\,.$$

Here F_t is a nondecreasing left-continuous operator function which tends to zero as $t \to -\infty$ and satisfies the condition

$$(11) \qquad (F_t f, f) \leqq (f, f) \qquad (-\infty < t \leqq \infty).$$

In contrast with Section 65, the operator function S_λ does not satisfy the Hilbert functional equation. Hence, we cannot derive the orthogonality relation

$$(12) \qquad F_u F_v = F_s \qquad (s = \min \{u, v\}).$$

Therefore, we fail to have the relation used in the end of Section 65 in the proof of the property

$$(13) \qquad \lim_{t \to \infty} F_t f = f \qquad (f \in \mathrm{H}).$$

In order to prove that the operator function F_t is a generalized resolution of the identity, we must prove property (13), without the aid of (12).

According to the lemma on monotone increasing sequences of operators (Section 85), the limit on the left side of equation (13) exists. Hence, it is sufficient to show that as $t \to \infty$ the operator F_t tends weakly to E from the left, i.e., that

$$\lim_{t \to \infty} (F_t f, g) = (f, g)$$

for all $f, g \in \mathrm{H}$. Evidently this relation holds if and only if

$$(14) \qquad \lim_{t \to \infty} (F_t f, f) = (f, f)$$

for all $f \in \mathrm{H}$. Since, by (11), the norm of the operator function F_t is bounded ($\| F_t \| \leqq 1$), it is sufficient to verify equation (14) for some dense set of vectors in H. As such a set we take the domain D_A of the operator A.

By (10), equation (14) is equivalent to the equation

$$(15) \qquad \lim_{\eta \to \infty} i\eta (S_{i\eta} f, f) = -(f, f)$$

for all $f \in \mathrm{H}$. Since the point $(S_{i\eta} f, f)$ lies inside the circle $C(f; \lambda)$ it is evident that

$$| i\eta (S_{i\eta} f, f) + (f, f) | \leqq \max_{-\infty < \tau \leqq \infty} | i\eta (R^\tau_{i\eta} f, f) + (f, f) |.$$

Here R^τ_λ denotes the resolvent of type I which corresponds to the parameter τ. It remains to prove that

$$| i\eta (R^\tau_{i\eta} f, f) + (f, f) |$$

tends to zero uniformly with respect to τ $(-\infty < \tau \leqq \infty)$. This is true since, for $f \in \mathrm{D}_A$ and $\eta > 0$,

$$| i\eta (R^\tau_{i\eta} f, f) + (f,f) | = \left| \int_{-\infty}^{\infty} \frac{i\eta}{t - i\eta} \, d(E^\tau_t f, f) + \int_{-\infty}^{\infty} d(E^\tau_t f, f) \right| =$$

$$= \left| \int_{-\infty}^{\infty} \frac{t}{t - i\eta} \, d(E^\tau_t f, f) \right| \leqq \sqrt{\int_{-\infty}^{\infty} t^2 d(E^\tau_t f, f)} \sqrt{\int_{-\infty}^{\infty} \frac{1}{|t - i\eta|^2} d(E^\tau_t f, f)} \leqq$$

$$\leqq \frac{1}{\eta} \| Af \| \cdot \| f \|.$$

Thus, relation (13) is proved so that the operator function F_t is indeed a generalized resolution of the identity and according to the theorem of Naimark it has the representation

$$F_t = P^+ E_t^+ .$$

Let us introduce the self-adjoint operator

$$A^+ f = \int_{-\infty}^{\infty} t \, d \, E_t^+ f.$$

Evidently

$$S_\lambda = P^+ R_\lambda^+ ,$$

where R_λ^+ is the resolvent of the operator A^+. To complete the proof of the theorem it remains to show that A^+ is an extension of A:

$$A^+ \supset A,$$

or that

$$R_\lambda^+ f = R_\lambda f$$

for $f \in \mathfrak{M}_\lambda$.

From formula (9′) it follows that

$$P^+ R_\lambda^+ f = S_\lambda f = R_\lambda f$$

for $f \in \mathfrak{M}_\lambda$. Therefore,

$$R_\lambda^+ f = R_\lambda f + h,$$

where $h \perp \mathrm{H}$. We show that $h = 0$. For $f \in \mathfrak{M}_\lambda$,

$$A R_\lambda f = f + \lambda \, R_\lambda f,$$
$$A^+(R_\lambda f + h) = f + \lambda \, R_\lambda f + \lambda \, h.$$

Letting $g = R_\lambda f$ we obtain

$$(A^+(g + h), g + h) = (Ag + \lambda h, g + h) = (Ag, g) + \lambda (h, h),$$

which is possible only for $h = 0$, since $\Im \lambda \neq 0$.

Hence, the theorem is completely proved.

5. Quasi-Self-Adjoint Extensions and the Characteristic Function of a Symmetric Operator

In the present section we consider another class of extensions of symmetric operators with finite and equal deficiency numbers which was introduced by Lifschitz.[7]

A *quasi-self-adjoint extension* of a symmetric operator A with the deficiency indices (m, m) $(m < \infty)$ is an arbitrary linear operator B which satisfies the conditions

$$(1) \qquad A \subset B \subset A^*,$$

$$(2) \qquad \dim \mathrm{D}_B = m \qquad (\mathrm{mod}\ \mathrm{D}_A),$$

but is not a self-adjoint extension of the operator A.

For simplicity we restrict ourselves to the case of operators with deficiency indices (1, 1). In this case condition (2) is a consequence of condition (1) and can be omitted. We shall assume that the operator A is simple (Section 81).

Next we present an example of a quasi-self-adjoint extension of a symmetric operator. Let P be the operator of differentiation in the space $L^2(0, a)$ with the boundary condition

$$\varphi(0) = \varphi(a) = 0.$$

In Chapter IV we proved that the domain of an arbitrary self-adjoint extension P_θ of the operator P is determined by the boundary condition

$$(3) \qquad \varphi(a) = \theta \varphi(0) \qquad (|\theta| = 1),$$

and, conversely, each such condition with $|\theta| = 1$ determines the domain of a self-adjoint extension of the operator P.

If in equation (3) θ is replaced by an arbitrary complex number ρ such that $|\rho| \neq 1$, then the operator P_ρ $(P_\rho \subset P^*)$ which is defined on set of the functions $\varphi(t)$ which satisfy the condition

$$(4) \qquad \varphi(a) = \rho \varphi(0) \qquad (|\rho| \neq 1)$$

is a quasi-self-adjoint extension of the operator P. It is evident that each quasi-self-adjoint extension of the operator P is given by condition (4) (one of them with $\rho = \infty$).

For quasi-self-adjoint extensions B of an operator A it is possible to introduce the Cayley transform S which is defined on the manifold

$$\mathrm{D}_s = \Delta_B(\lambda)$$

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[7] Cf. M. S. Lifschitz [2], [3], Vol. I.

by the formulas

$$(5)\qquad \varphi = (B - \lambda E)f$$

$$(f \in \mathrm{D}_B, \Im\lambda \neq 0)$$

$$(6)\qquad S\varphi = (B - \bar{\lambda}E)f.$$

This definition is ambiguous if the vector f, which is defined by the vector φ, is not unique, i.e., if λ is an eigenvalue of the operator B. But if λ is an eigenvalue of the operator B, then $\bar{\lambda}$ is not an eigenvalue, since otherwise

$$g_\lambda \in \mathrm{D}_B,\ g_{\bar{\lambda}} \in \mathrm{D}_B$$

and

$$B = A^*,$$

which would contradict condition (1).

Therefore, we can assume in formula (5) that λ is not an eigenvalue of the operator B. For definiteness we choose $\lambda = -i$ and then write formulas (5) and (6) in the form

$$(5')\qquad \varphi = (B + iE)f$$

$$(f \in \mathrm{D}_B)$$

$$(6')\qquad S\varphi = (B - iE)f.$$

It is evident that the operator S is defined in the whole space and is an extension of the Cayley transform V of the operator A. However, in contrast with the case of self-adjoint extensions, the operator S is not unitary.

We show that the orthogonal complement of the manifold

$$\mathrm{D}_V = \mathfrak{M}_{-i} = \Delta_A(-i)$$

is transformed by the operator S either into the orthogonal complement of the manifold

$$\Delta_V = \mathfrak{M}_i = \Delta_A(i),$$

or into the zero manifold. Indeed, if

$$g \perp \mathfrak{M}_{-i}$$

and

$$g = (B + iE)\,h \qquad (h \in \mathrm{D}_B),$$

then for each $f \in \mathrm{D}_A$

$$((B + iE)\,h, (A + iE)f) = 0.$$

On the other hand, using the inclusion $B \subset A^*$, we get for each $f \in \mathrm{D}_A$

$$\begin{aligned}(Sg, (A-iE)f) &= ((B-iE)h, (A-iE)f) = \\ &= (Bh, Af) - i(h, Af) + i(Bh, f) + (h, f) = \\ &= (Bh, Af) - i(Bh, f) + i(h, Af) + (h, f) = \\ &= ((B+iE)h, (A+iE)f) = 0,\end{aligned}$$

so that $Sg \perp \mathfrak{M}_i$ (in particular, it is possible that $Sg = 0$).

If g_1 denotes a vector orthogonal to the manifold $\mathfrak{M}_i$ such that $\|g_1\| = \|g\|$, then, on the basis of the proof,

$$Sg = \kappa g_1, \tag{7}$$

where the number κ does not have absolute value one, since otherwise the operator B, which obviously is expressed in terms of S by means of the formulas

$$f = \tfrac{1}{2i}(E - S)\varphi, \tag{8}$$

$$(\varphi \in \mathrm{H})$$

$$Bf = \tfrac{1}{2}(E + S)\varphi \tag{9}$$

would be self-adjoint.

The operator $S \equiv S_\kappa$ is called a *quasi-unitary extension* of the isometric operator V. It is easy to verify that $S_\kappa^* = S_{\bar{\kappa}}$.

In general, we define a quasi-unitary extension of a given isometric operator V with deficiency indices (m, m) $(m < \infty)$ as an arbitrary linear but not unitary operator $S \supset V$ which is defined in the whole space and which transforms the orthogonal complement of the manifold D_V into a sub-space orthogonal to Δ_V (or into the null element).

It is evident that each quasi-unitary extension S of the operator V generates by formulas (8) and (9) a quasi-self-adjoint extension B of the operator A.

For the case of deficiency indices $(1, 1)$ there exists a one-to-one correspondence between the set of all self-adjoint extensions B of the operator A (respectively of quasi-unitary extensions S of the operator V) and the set of all complex numbers κ not equal to one in absolute value. In view of this relation, we denote the corresponding extension by B_κ (respectively, S_κ).

We turn now to the study of the spectrum of a quasi-self-adjoint extension B of the operator A with deficiency indices $(1, 1)$. We recall that by the general definition of Section 43, the number λ is called a regular point of the linear operator T if the operators $(T - \lambda E)^{-1}$ exists, is bounded, and is defined in the whole space. The spectrum of the operator T is defined as the complement of the set of its regular points.

THEOREM 1: *The spectrum of a quasi-self-adjoint extension B of a simple operator A with deficiency indices* (1, 1) *consists of the spectral kernel* (*cf. Section* 83) *of the operator A and the eigenvalues. The set of the eigenvalues lies in its entirety either in the upper or in the lower half-plane. We omit the special case*[8] *when the whole half-plane* (*upper or lower*) *consists of eigenvalues, and, as a consequence, the set of eigenvalues can have only real limit points in addition to the eigenvalues themselves.*

Proof: We denote the spectral kernel of the operator A by Λ. For each $\lambda \in \Lambda$, the operator $(A-\lambda E)^{-1}$ is unbounded and hence it is not possible for the operator $(B-\lambda E)^{-1}$ to be bounded. Hence Λ is contained in the spectrum of the operator B.

We assume now that $\lambda \notin \Lambda$. If the operator $(B-\lambda E)^{-1}$ does not exist, then λ is an eigenvalue of the operator B. If the operator $(B-\lambda E)^{-1}$ exists then λ is a regular point of the operator B. Indeed the operator $(B-\lambda E)^{-1}$ cannot be unbounded, since the operator $(A-\lambda E)^{-1}$ is bounded and $\Delta_B(\lambda)$ differs from $\Delta_A(\lambda)$ by not more than one dimension. It remains to show that $\Delta_B(\lambda) = \mathrm{H}$. Assuming the contrary, we find that $\Delta_B(\lambda) = \Delta_A(\lambda)$. Now if f is a vector in D_B which does not belong to D_A, then the vector

$$(B-\lambda E)f = f^*,$$

which belongs to $\Delta_B(\lambda) = \Delta_A(\lambda)$, can be represented in the form

$$f^* = (A-\lambda E)f' = (B-\lambda E)f' \qquad (f' \in \mathrm{D}_A).$$

Hence

$$(B-\lambda E)(f-f') = 0,$$

which contradicts the existence of the operator $(B-\lambda E)^{-1}$. Thus, it is proved that λ is a regular point, and the spectrum of the operator B consists of the spectral kernel Λ of the operator A and the eigenvalues.

We introduce the Cayley transform S_κ of the operator $B = B_\kappa$, and then we notice without difficulty that the point λ runs through the spectrum of the operator B as the point

$$\zeta = \frac{\lambda - i}{\lambda + i} \tag{10}$$

runs through the spectrum of the operator S_κ, and conversely. Moreover, formula (10) establishes a one-to-one correspondence between the eigenvalues of the operator B and of the operator S_κ.

On the basis of what has been said, it is sufficient to determine the eigenvalues of the operator S_κ. Since the absolute values of the eigenvalues of the operators S_κ are greater or less than one according as $|\kappa| > 1$ or $|\kappa| < 1$, all eigenvalues of the operator B lie in the upper half-plane (if $|\kappa| < 1$) or in the lower half-plane (if $|\kappa| > 1$).

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[8] This case will be treated later (cf. the corollary of Theorem 3).

We assume for definiteness that $|\kappa| < 1$. In what follows g denotes a unit vector orthogonal to the manifold $\mathfrak{M}_i$, $\mathring{U}$ denotes a unitary extension of U and g^* is the vector defined by

$$g^* = \mathring{U}g.$$

We represent the eigenvector of the operator S_κ which corresponds to the number ζ, in the form $\varphi + \alpha g$ with $\varphi \in \mathrm{D}_V$. Thus,

$$S_\kappa(\varphi + \alpha g) = \zeta(\varphi + \alpha g).$$

Then

$$S_\kappa\varphi + \alpha\kappa g^* = \zeta\varphi + \alpha\zeta g,$$

or

$$\mathring{U}\varphi - \zeta\varphi = \alpha(\zeta g - \kappa g^*),$$

and this yields

$$\frac{1}{\alpha}\varphi = \zeta(\mathring{U} - \zeta E)^{-1}g - \kappa(\mathring{U} - \zeta E)^{-1}g^*. \tag{11}$$

We form the scalar product of both members of (11) with g and obtain an equation satisfied by the eigenvalues of the operator S_κ:

$$\zeta((\mathring{U} - \zeta E)^{-1}g, g) - \kappa((\mathring{U} - \zeta E)^{-1}g^*, g) = 0. \tag{12}$$

Reversing the order of calculations we find that every root of equation (12) is an eigenvalue of the operator S_κ.

Since the left member of equation (12) is a regular function for $|\zeta| \neq 0$, the assertion of the theorem concerning the limit points of the discrete spectrum of the operator B is proved with the exception of the case when the left side of equation (12) becomes zero for $|\zeta| < 1$.

In this last case each point ζ inside the unit circle is an eigenvalue of the operator S_κ.

We show now that the special case mentioned in the formulation of the theorem can occur. Let H be a separable Hilbert space with an orthonormal basis $\{e_k\}_{-\infty}^{\infty}$ and let $\tilde{U}$ be the unitary operator defined by the formulas

$$\tilde{U}e_k = e_{k-1}.$$

Further, let $\tilde{V} \subset \tilde{U}$ and $\mathrm{D}_{\tilde{V}} \perp e_0$ and, hence, $\Delta_{\tilde{V}} \perp e_{-1}$. The operator $\tilde{V}$ is an isometric operator with deficiency indices (1, 1).

Now let S_0 be the quasi-unitary extension of the operator $\tilde{V}$ defined by the condition

$$S_0 e_0 = 0.$$

It is easy to verify that for each ζ inside the unit circle, the vector

$$\sum_{k=0}^{\infty} \zeta^k e_k$$

is an eigenvector of the operator S which corresponds to the eigenvalue ζ.

Later, we shall see that this special case is unique in the following sense. A simple symmetric operator with deficiency indices (1, 1) having a quasi-self-adjoint extension with a point spectrum which forms an entire half-plane is isomorphic to the Cayley transform of the operator $\tilde{V}$.

We turn now to the transformation of equation (12). If $\mathring{F}_t$ is the spectral function of the operator $\mathring{U}$, then

$$2\zeta((\mathring{U}-\zeta E)^{-1}g, g) = 2\zeta \int_0^{2\pi} \frac{1}{e^{is}-\zeta} d(\mathring{F}_s g, g) = \tag{13}$$

$$= \int_0^{2\pi} \frac{e^{is}+\zeta}{e^{is}-\zeta} d(\mathring{F}_s g, g) - 1,$$

and

$$2((\mathring{U}-\zeta E)^{-1}g^*, g) = 2((\mathring{U}-\zeta E)^{-1}\mathring{U}g, g) = \tag{14}$$

$$= 2\int_0^{2\pi} \frac{e^{is}}{e^{is}-\zeta} d(\mathring{F}_s g, g) = \int_0^{2\pi} \frac{e^{is}+\zeta}{e^{is}-\zeta} d(\mathring{F}_s g, g) + 1.$$

From the last formulas it follows in particular that the scalar product $((\mathring{U}-\zeta E)^{-1}g^*, g)$ does not vanish for $|\zeta| < 1$. Hence, equation (12) can be represented in the form

$$w(\zeta) - \kappa = 0,$$

where

$$w(\zeta) = \frac{\zeta((\mathring{U}-\zeta E)^{-1}g, g)}{((\mathring{U}-\zeta E)^{-1}g^*, g)} \qquad (g^* = \mathring{U}g). \tag{15}$$

On the basis of formulas (13) and (14) the function $w(\zeta)$ can be represented in the form

$$w(\zeta) = \frac{\Phi(\zeta)-1}{\Phi(\zeta)+1}, \tag{15a}$$

where

$$\Phi(\zeta) = \int_0^{2\pi} \frac{e^{is}+\zeta}{e^{is}-\zeta}\, d(\mathring{F}_s g, g). \tag{15b}$$

From these representations it follows (Section 59) that the function $w(\zeta)$ is regular in the unit circle, maps it into itself and satisfies the normalization condition $w(0) = 0$.

Following Lifschitz we call the function $w(\zeta)$ *the characteristic function of the isometric operator* V and the function

$$\omega(\lambda) = w\left(\frac{\lambda - i}{\lambda + i}\right) \tag{15c}$$

the characteristic function of the symmetric operator A.

The function $\omega(\lambda)$ is regular in the upper half-plane, maps it into a subset of the unit circle, and satisfies the normalization condition $\omega(i) = 0$.

In order to justify the above definitions, it is necessary to show that the function $w(\zeta)$ is defined in principle by the operator V although (cf. formula (15)) it depends formally on the choice of the unitary extension $\mathring{U}$. With this aim we express the operator $\mathring{U}$ in formula (15) in terms of its Cayley transform:

$$\mathring{U} = (\mathring{A} - iE)(\mathring{A} + iE)^{-1}.$$

After an elementary calculation we obtain for the characteristic function $\omega(\lambda)$ the formula

$$\omega(\lambda) = \frac{\lambda - i}{\lambda + i}\, \frac{(\{E + (\lambda - i)\mathring{R}_\lambda\} g, g^*)}{(\{E + (\lambda - i)\mathring{R}_\lambda\} g, g)}. \tag{16}$$

Here $\mathring{R}_\lambda$ is the resolvent of the operator $\mathring{A}$.

In Section 84 it was established that the vector

$$\{E + (\lambda - i)\mathring{R}_\lambda\} g$$

belongs to the deficiency subspace $\mathfrak{N}_{\bar{\lambda}}$, which is one-dimensional in the case under consideration. Hence, for another choice of the extension $\mathring{A}$ this vector is replaced by a scalar multiple of itself. And this does not alter $\omega(\lambda)$. As far as the vector g^* is concerned, it is multiplied by a number with absolute value one.

Thus, the characteristic function of a symmetric (isometric) operator is determined by this operator to within a multiplicative constant with absolute value one. We do not distinguish between two characteristic functions which differ from one another by such a factor.

Now formula (16) can be written in the form

$$\omega(\lambda) = \frac{\lambda - i}{\lambda + i} \, \frac{(g_\lambda, g^*)}{(g_\lambda, g)}, \tag{17}$$

where g_λ is an arbitrary solution of the equation

$$A^*h - \lambda h = 0.$$

As an example, we calculate the characteristic function of the operator P of differentiation on the interval $[0, a]$ with the boundary conditions

$$\varphi(0) = \varphi(a) = 0.$$

In this case

$$g = \frac{\sqrt{2}}{\sqrt{e^{2a} - 1}} e^t, \quad g^* = \frac{\sqrt{2}}{\sqrt{1 - e^{2a}}} e^{-t}, \quad g_\lambda = e^{-i\lambda t},$$

and, by formula (17),

$$\omega(\lambda) = \frac{e^a - e^{-ia\lambda}}{1 - e^a e^{-ia\lambda}} \, . \tag{18}$$

In addition to $w(\zeta)$ and $\omega(\lambda)$ we introduce the functions

$$w(\zeta; \kappa) = \frac{w(\zeta) - \kappa}{\bar{\kappa} w(\zeta) - 1},$$

$$\omega(\lambda; \kappa) = \frac{\omega(\lambda) - \kappa}{\bar{\kappa}\omega(\lambda) - 1},$$

and call them the *characteristic functions* of the quasi-unitary extension S of the operator V and, correspondingly, of the quasi-self-adjoint extension B_κ of the operator A. The characteristic functions $w(\zeta; \kappa)$ and $\omega(\lambda; \kappa)$ are normed by the conditions

$$w(0; \kappa) = \kappa, \qquad \omega(i; \kappa) = \kappa.$$

With the aid of the characteristic function $\omega(\lambda)$ the spectral kernel of the operator A is defined, as shown in the following proposition.

THEOREM 2: *In order that the real number* λ_0 *be a point of regular type of a simple symmetric operator* A *with deficiency indices* (1, 1) *it is necessary and sufficient that both of the following conditions be satisfied:*

1° *the function* $\omega(\lambda)$ *is regular in a neighborhood of* λ_0;

2° $|\omega(\lambda)| = 1$ in some interval $\lambda_0 - \epsilon < \lambda < \lambda_0 + \epsilon$.

Proof: Let λ_0 be a point of regular type of the operator A and V the Cayley transform of A. It is evident that there exists a self-adjoint extension $\mathring{A} \supset A$ for which the point λ_0 is regular, because if two self-adjoint extensions $\overset{\circ}{A}_1$ and $\overset{\circ}{A}_2$ have the same eigenvalue λ_0, then

$$\overset{\circ}{A}_1 f_1 = \lambda_0 f_1,\ \overset{\circ}{A}_2 f_2 = \lambda_0 f_2 \quad (f_1, f_2 \notin \mathrm{D}_A).$$

Then

$$A^* f_1 = \lambda_0 f_1,\ A^* f_2 = \lambda_0 f_2,$$

and hence

$$f_1 = \alpha f_2,$$

so that

$$\mathrm{D}_{\overset{\circ}{A}_1} = \mathrm{D}_{\overset{\circ}{A}_2}$$

and

$$\overset{\circ}{A}_1 = \overset{\circ}{A}_2.$$

In formula (15) let $\overset{\circ}{U}$ be the Cayley transform of $\overset{\circ}{A}$. Then from formulas (15a), (15b), (15c) it follows that the function $\omega(\lambda)$ has properties 1° and 2°.

Conversely, let $\omega(\lambda)$ have properties 1° and 2°. Then choosing the extension $\overset{\circ}{U}$ in formula (15) such that

$$w(\zeta_0) \neq 1. \qquad \left(\zeta_0 = \frac{\lambda_0 - i}{\lambda_0 + i}\right),$$

the function

$$\Phi(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)}$$

is regular in a neighborhood of the point $\zeta_0 = e^{is_0}$. Furthermore, on some arc of the unit circle containing the point ζ_0, $|\Phi(\zeta)| = 1$. Therefore, after the application of the inversion formula to the representation (15b) it follows that ζ_0 is a point of constancy of the function $(\overset{\circ}{F}_s g, g)$ and, since the vector g generates the operator $\overset{\circ}{U}$, ζ_0 is a regular point of this operator (cf. Section 69). Thus, λ_0 is a regular point of the operator $\overset{\circ}{A}$ and, hence, does not belong to the spectral kernel of A.

From the proof of Theorem 2 we obtain the following refinement of Theorem 1: *all finite limit points of the set of eigenvalues of any quasi-self-adjoint extension B_κ of the given operator A belongs* (*with the exception of the case mentioned in Theorem* 1) *to the spectral kernel of the operator A.*

If $\zeta_k (k = 1, 2, 3, \ldots)$ are the roots of the characteristic function $w(\zeta; \kappa)$ of a quasi-unitary extension S_κ of the operator V for $\kappa \neq 0$, then $w(\zeta; \kappa)$ can be represented in the form

$$w(\zeta; \kappa) = e^{-G(\zeta)} \prod_{k=1}^{\infty} \frac{\zeta_k - \zeta}{1 - \bar{\zeta}_k \zeta} \frac{|\zeta_k|}{\zeta},$$

where $G(\zeta)$ is a regular function with non-negative real part for $|\zeta| < 1$. If we represent the function $G(\zeta)$ in the form (Section 69)

$$G(\zeta) = \int_0^{2\pi} \frac{e^{is} + \zeta}{e^{is} - \zeta}\, d\rho(s) + i\beta,$$

where $\rho(s)$ is a nondecreasing function of bounded variation then we obtain

$$(19) \quad w(\zeta; \kappa) = Ce^{\int_0^{2\pi} \frac{\zeta + e^{is}}{\zeta - e^{is}} d\rho(s)} \prod_{k=1}^{\infty} \frac{1 - \frac{\zeta}{\zeta_k}}{1 - \bar{\zeta}_k \zeta} |\zeta_k| \quad (|C| = 1).$$

From Theorems 1 and 2, it follows that the point spectrum of a quasi-unitary extension S_κ of the operator V consists of the points ζ_k and the rest of the spectrum consists of points of increase of the function $\rho(s)$ and limit points of the set of roots ζ_k.[9]

If the spectral kernel of the operator is empty, then the eigenvalues of an arbitrary quasi-self-adjoint extension B_κ of the operator A have no finite limit points.

In particular, it can happen that the spectrum of some quasi-self-adjoint extension of the operator A is void. An example is the quasi-self-adjoint extension of the differential operator P with the boundary condition

$$\varphi(0) = 0.$$

Later (cf. Theorem 4) we see that this special case is unique if one does not consider isomorphic operators as distinct.

We prove now a general theorem which shows that the characteristic function determines an operator up to an isomorphism.

THEOREM 3: *In order that simple symmetric (isometric) operators with deficiency indices* (1, 1) *be unitary equivalent, it is necessary and sufficient that their characteristic functions coincide.*

Proof: We give the proof for isometric operators. Let the operators V and $\tilde{V}$ which are defined in the spaces H and $\tilde{\mathrm{H}}$ respectively be unitary equivalent, i.e., let

$$\tilde{V} = \mathfrak{U} V \mathfrak{U}^{-1},$$

where $\mathfrak{U}$ is an isometric operator which maps H on $\tilde{\mathrm{H}}$. Taking some unitary extension U of the operator V we construct a unitary extension $\tilde{U}$ of the operator $\tilde{V}$ by the formula

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[9] Starting from the representation (19) for the characteristic function $w(\zeta; x)$ M. S. Lifschitz investigated the question of the invariant subspaces of the operators S_κ (cf. M. S. Lifschitz (4), Vol. I).

$$\tilde{U} = \mathfrak{U}\, U\, \mathfrak{U}^{-1}.$$

Furthermore, we choose a unit vector $g \in \mathrm{H}$ orthogonal to the manifold D_V and we let

$$\tilde{g} = \mathfrak{U} g.$$

For these choices of the vector $\tilde{g}$ and of the operator $\tilde{U}$, the formulas

$$w(\zeta) = \frac{\zeta((U - \zeta E)^{-1} g, g)_H}{((U - \zeta E)^{-1} U g, g)_H}, \tag{20}$$

$$\tilde{w}(\zeta) = \frac{\zeta((\tilde{U} - \zeta \tilde{E})^{-1} \tilde{g}, \tilde{g})_{\tilde{H}}}{((\tilde{U} - \zeta \tilde{E})^{-1} \tilde{U} \tilde{g}, \tilde{g})_{\tilde{H}}} \tag{21}$$

imply that

$$\tilde{w}(\zeta) = w(\zeta).$$

Conversely, let the characteristic functions $\tilde{w}(\zeta)$ and $w(\zeta)$ of the operators $\tilde{V}$ and V, which are defined by formulas (20) and (21), coincide. Let

$$\tilde{\Phi}(\zeta) = \frac{1 + \tilde{w}(\zeta)}{1 - \tilde{w}(\zeta)}, \quad \Phi(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)},$$

and apply the inversion formula to the representations of the functions $\tilde{\Phi}(\zeta)$ and $\Phi(\zeta)$ to obtain

$$(\tilde{E}_t \tilde{g}, \tilde{g})_{\tilde{H}} = (E_t g, g)_H.$$

On the other hand, the simplicity of the operators $\tilde{V}$ and V implies the simplicity of the spectra of their unitary extensions $\tilde{U}$ and U; here the deficiency vectors $\tilde{g}$ and g are the generating elements for $\tilde{U}$ and U (Section 81).

Thus, the operators $\tilde{U}$ and U are reduced to the same canonical form—to the operator of multiplication by e^{it} in the space L_σ^2 with the distribution function

$$\sigma(t) = (\tilde{E}_t \tilde{g}, \tilde{g})_{\tilde{H}} = (E_t g, g)_H.$$

and, hence, are isomorphic (Section 69). From the isomorphism of the operators $\tilde{U}$ and U it immediately follows that the operators $\tilde{V}$ and V are isomorphic. Thus, the theorem is proved.

REMARK: Each simple symmetric operator with deficiency indices (1, 1) which has a quasi-self-adjoint extension with point spectrum which covers an entire half-plane is isomorphic to the Cayley transform of the operator $\tilde{V}$ defined on page 151.

Indeed the characteristic function of such an operator must be identically zero, so that all such operators are isomorphic.

The following theorem gives us an interesting abstract characterization of the operator of differentiation on a finite interval.

THEOREM 4: *Each simple symmetric operator with deficiency indices* (1, 1) *which admits a quasi-self-adjoint extension without spectrum is isomorphic to the operator of differentiation on a finite interval.*

Proof: Let A be an operator with the property indicated in the formulation of the theorem, let B_κ be a self-adjoint extension without spectrum and let $\omega(\zeta; \kappa)$ be the characteristic function of the operator B_κ. Furthermore, let V be the Cayley transform of the operator A and let $w(\zeta; \kappa)$ be the characteristic function of the Cayley transform S_κ of the operator B_κ.

Since B_κ has no eigenvalues, the function $w(\zeta; \kappa)$, which maps the unit circle on a subset of itself, never vanishes. Therefore,

$$w(\zeta; \kappa) = e^{-G(\zeta)},$$

where the function $G(\zeta)$ is regular in the unit circle and has there a nonnegative real part. Replace ζ by

$$\frac{\lambda - i}{\lambda + i}$$

to obtain

$$\omega(\lambda; \kappa) = e^{iH(\lambda)}.$$

Here $H(\lambda)$ is a function which is regular in the upper half-plane and maps it into a subset of itself.

According to Section 59, the function $H(\lambda)$ can be represented in the form

$$H(\lambda) = \alpha + \mu\lambda + \int_{-\infty}^{\infty} \frac{1 + i\lambda}{t - \lambda} d\sigma(t),$$

where α is real, $\mu \geqq 0$ and $\sigma(t)$ is a nondecreasing function of bounded variation.

Since the spectral kernel of the operator A is empty, the function $H(\lambda)$ is also regular and real on the entire real axis. Therefore, by the Stieltjes inversion formula (Section 59)

$$\sigma(t) = \text{const.},$$

and

$$H(\lambda) = \alpha + \mu\lambda.$$

Thus, except possibly for a constant factor of absolute value one,

$$\omega(\lambda; \kappa) = e^{i\mu\lambda}.$$

This yields

$$\omega(\lambda) = \frac{\kappa - \omega(\lambda; \kappa)}{1 - \bar{\kappa}\,\omega(\lambda; \kappa)} = \frac{\kappa - e^{i\mu\lambda}}{1 - \bar{\kappa}e^{i\mu\lambda}}\,.$$

But since

$$\omega(i) = 0,$$

we have

$$\omega(\lambda) = \frac{e^{\mu} - e^{-i\mu\lambda}}{1 - e^{\mu}e^{-i\mu\lambda}}.$$

Comparing this formula with formula (18) we see that the characteristic function of the operator A coincides with the characteristic function of the operator of differentiation on the interval $[0, \mu]$. By Theorem 3 the operator A is unitarily equivalent to the operator of differentiation, which was to be proved.

In connection with Theorem 3, there arises the question of the existence of a symmetric (isometric) operator with given characteristic function. For isometric operators this question can be answered in the affirmative:

THEOREM 5: *Let $w(\zeta)$ be a function which is regular in the unit circle, maps it in a subset of itself and satisfies the normalization condition $w(0)=0$. Then there exists an isometric operator V for which $w(\zeta)$ is the characteristic function.*

Proof: We define a function $\Phi(\zeta)$ by means of the equation

$$w(\zeta) = \frac{\Phi(\zeta) - 1}{\Phi(\zeta) + 1}\,.$$

By Section 59, $\Phi(\zeta)$ can be represented in the form

$$\Phi(\zeta) = \int_0^{2\pi} \frac{e^{is} + \zeta}{e^{is} - \zeta}\, d\sigma(s),$$

where $\sigma(s)$ is a nondecreasing function with total variation one.

We introduce the space $L^2_\sigma(0, 2\pi)$ and define in it the unitary operator of multiplication by e^{it}:

$$\overset{\circ}{U}f(t) = e^{it}f(t).$$

Let V be the isometric operator which coincides with the operator $\overset{\circ}{U}$ on the hyperplane D_V orthogonal to the function $g(t) \equiv 1$. It is easy to verify that V satisfies the conditions of the theorem.

Indeed, since the spectral function $\overset{\circ}{E}_t$ of the operator $\overset{\circ}{U}$ is defined by the equation

$$\overset{\circ}{E}_t f(s) = \begin{cases} f(s) & (s \leqq t), \\ 0 & (s > t), \end{cases}$$

we have

$$((\overset{\circ}{U} - \zeta E)^{-1} g, g) = \int_0^{2\pi} \frac{d\sigma(s)}{e^{is} - \zeta},$$

$$((\overset{\circ}{U} - \zeta E)^{-1} \overset{\circ}{U} g, g) = \int_0^{2\pi} \frac{e^{is} d\sigma(s)}{e^{is} - \zeta},$$

and

$$\frac{\zeta((\overset{\circ}{U} - \zeta E)^{-1} g, g)}{((\overset{\circ}{U} - \zeta E)^{-1} \overset{\circ}{U} g, g)} = \frac{\Phi(\zeta) - 1}{\Phi(\zeta) + 1} = w(\zeta),$$

which proves the theorem (cf. formula (15)).

If, for an isometric operator V with a given characteristic function $w(\zeta)$, the manifold $\Delta_V(1)$ is not dense in H, then it is impossible to pass from V to its Cayley transform A. Therefore, in this case there exists no symmetric operator with characteristic function

$$\omega(\lambda) = w\left(\frac{\lambda - i}{\lambda + i}\right). \tag{22}$$

In order that the manifold $\Delta_V(1)$ be dense in H it is necessary to impose an additional restriction on the function (22). This matter is resolved in the following theorem.

THEOREM 6: *Let $\omega(\lambda)$ be any function which is regular in the upper half-plane, maps it into the interior of the unit circle and satisfies the normalization condition $\omega(i) = 0$. Then $\omega(\lambda)$ is the characteristic function of some simple symmetric operator if and only if*

$$\lim_{\lambda \to \infty} \lambda\{\omega(\lambda) - e^{i\alpha}\} = \infty \qquad (0 < \epsilon \leqq \arg \lambda \leqq \pi - \epsilon)$$

for each α such that $0 \leqq \alpha < 2\pi$.

Proof: We show that the manifold $\Delta_V(1)$ is not dense in H if and only if there exists a unitary extension of the operator V with an eigenvalue equal to one.

The sufficiency of the conditon is evident since U is a unitary extension of the operator V and $U\psi = \psi$ $(\psi \neq 0)$, so that for each $\varphi \in D_V$,

$$((V - E)\varphi, \psi) = ((U - E)\varphi, \psi) = (\varphi, (U^{-1} - E)\psi) = 0.$$

For the proof of the necessity of the condition we assume that

$$((V-E)\varphi, \psi)=0 \qquad (\psi \neq 0)$$

for each $\varphi \in D_V$. We represent each vector $f \in H$ in the form $f = \varphi + \gamma g$, and obtain for an arbitrary quasi-unitary extension S_κ of the operator V the equation

$$(23) \quad ((S_\kappa - E)f, \psi) = (V\varphi - \varphi + \gamma\kappa g^* - \gamma g, \psi) = \gamma\{\kappa(g^*,\psi) - (g,\psi)\}.$$

Note that $(g^*, \psi) \neq 0$, since otherwise the vector ψ would have the representation $\psi = V\varphi_1 (\varphi_1 \in D_V,\ \varphi_1 \neq 0)$ so that

$$((V-E)\varphi, V\varphi_1) = 0 \qquad (\varphi \in D_V),$$

and, consequently,

$$(\varphi, (E-V)\varphi_1) = 0.$$

It follows that

$$V\varphi_1 - \varphi_1 = \gamma_1 g,$$

or

$$V\varphi_1 = \varphi_1 + \gamma_1 g.$$

But the orthogonality of the summands on the right side of the last equation and the condition $\| V\varphi_1\| = \|\varphi_1\|$ imply that $V\varphi_1 = \varphi_1 (\varphi_1 \neq 0)$, which contradicts the fact that the operator V is simple. Thus $(g^*, \psi) \neq 0$ and we can substitute

$$\kappa = \frac{(g, \psi)}{(g^*, \psi)}$$

in (23). It follows that, for each $f \in H$,

$$((S_\kappa - E)f, \psi) = 0,$$

or

$$(f, (S_\kappa^* - E)\psi) = 0.$$

Since $S_\kappa^* = S_{\bar{\kappa}}$, this equation implies that $S_{\bar{\kappa}}\psi = \psi\ (\psi \neq 0)$ which is possible only for $|\kappa| = 1$. Hence $S_{\bar{\kappa}}$ is the desired unitary extension of the operators V with an eigenvalue equal to one.

Now the proof of the theorem is reduced to the determination of necessary and sufficient conditions which the characteristic function of an isometric operator V must satisfy in order that none of its unitary extensions has an eigenvalue equal to one.

Let $\mathring{U}$ be the unitary extension of the operator V which appears in formula (15) and let $U^{(\alpha)}$ be an arbitrary unitary extension of the same operator such that $U^{(\alpha)}g = e^{i\alpha}g^*$. The deficiency element g is a generating element for each of the operators $U^{(\alpha)}$. Hence, the operator $U^{(\alpha)} (0 \leq \alpha < 2\pi)$

does not have an eigenvalue equal to one if and only if the distribution function $(F_t^{(\alpha)}g, g)$ has no jump at $t = 0$, where $F_t^{(\alpha)}$ is the spectral function of the operator $U^{(\alpha)}$ $(0 \leqq \alpha < 2\pi)$.

[10]On the other hand it follows from the formulas (15a), (15b), (15c) for the characteristic function $w(\zeta)$ that

$$\frac{e^{i\alpha}}{e^{i\alpha} - w(\zeta)} = \int_0^{2\pi} \frac{e^{it}}{e^{it} - \zeta}\, d(F_t^{(\alpha)}g, g).$$

This equation yields the value μ of the jump of the function $(F_t^{(\alpha)}g, g)$ at the point $t = 0$, namely

$$\mu = \lim_{\zeta \to 1} \frac{(1 - \zeta)\, e^{i\alpha}}{e^{i\alpha} - w(\zeta)}.$$

Using the indicated derivations, it is not difficult to complete the proof of the theorem. This task is left to the reader.

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[10] Translator's Note: This sentence and the one following do not appear in the original (Russian) edition. They are taken from the authorized German edition.

Appendix II

DIFFERENTIAL OPERATORS

1. Self-Adjoint Differential Expressions

We begin with the presentation of some facts concerning real differential expressions (or, let us agree to call them *differential operations*) which are self-adjoint in the sense of Lagrange.

In analysis courses it is established that a self-adjoint differential operation of the second order

$$l_0 D^2 + l_1 D + l_2 D^0 \quad \left(l_k = l_k(t),\ D^k = \frac{d^k}{dt^k}\right)$$

under the assumption of the k-fold differentiability of the coefficients $l_k(t)$ can be represented in the form

$$- Dp_0 D + D^0 p_1 D^0,$$

where $p_0(t)$ is differentiable. However, it is possible to consider such operations without the assumption of the differentiability of the function $p_0(t)$. In this case the operator can be applied only to differentiable functions $\varphi(t)$ such that the product $p_0\varphi'$ is absolutely continuous.

We turn now to self-adjoint differential expressions of order $2n$. We assume as the canonical form of such an operation the expression

$$(1) \quad l = p_n D^0 - D\{p_{n-1} D - D[p_{n-2} D^2 - \ldots - D(p_1 D^{n-1} - Dp_0 D^n) \ldots]\}.$$

If the coefficient $p_{n-k}(t)$ is k times ($k=0, 1, \ldots, n$) differentiable, then this is a self-adjoint ordinary differential operation, as may be verified immediately. Since we do not assume the differentiability of the coefficients, we call operation (1) a *quasi-differential operation.*

Let (a, b) be an open finite or infinite interval in which the differential operation (1) is considered. We assume that the coefficients $p_k(t)$ are measurable in this interval and satisfy the conditions

$$(2) \qquad \int_\alpha^\beta \frac{dt}{|p_0(t)|} < \infty, \int_\alpha^\beta |p_k(t)|\, dt < \infty, \ (k = 1, 2, \ldots, n)$$

in each closed subinterval $[\alpha, \beta]$ of (a, b).

If the interval (a, b) is finite and if condition (2) is satisfied for $\alpha=a$, $\beta=b$, then the operation (1) is called *regular.* If the interval (a, b) is infinite, or it is finite but conditions (2) are not satisfied for $\alpha=a$ or $\beta=b$, then the operation (1) is called *singular.* The left end point a is called *singular* if $a=-\infty$ or $a>-\infty$, but (2) fails with $\alpha=a$. Singularity of the right end point b is defined analogously. If an end point is not singular, then it is called *regular.* Without loss of generality, we consider singular differential operations with one singular end point on the semi-axis $(0, \infty)$ and with two singular end points on the entire axis $(-\infty, \infty)$.

For convenience we introduce, in place of the derivative $D^k\varphi = \varphi^{(k)}$, the so-called quasi-derivative $D^{[k]}\varphi = \varphi^{[k]}$ which is defined by the following formulas:

$$\begin{aligned} D^{[k]} &= D^k \qquad (k = 0,1,2,\ldots,n-1),\\ D^{[n]} &= p_0 D^n,\\ D^{[n+k]} &= p_k D^{n-k} - DD^{[n+k-1]} \qquad (k = 1,2,\ldots,n). \end{aligned}$$

Now operation (1) can be expressed in the form

$$l = D^{[2n]}.$$

Let D* denote the class of all functions $\varphi(t) \in L^2(a, b)$ for which each quasi-derivative $\varphi^{[k]}(t)$ $(k = 0, 1, \ldots, 2n-1)$ is absolutely continuous and the quasi-derivative $\varphi^{[2n]}(t)$ belongs to $L^2(a, b)$. It is evident that D* is the maximal linear manifold in $L^2(a, b)$, on which the operation l has a natural meaning and can be considered as an operator in $L^2(a, b)$. We denote this operator by L^*, so that $\mathrm{D}^* = \mathrm{D}_{L^*}$. We shall see later the purpose of this notation.

From (1) it is easy to obtain for each pair of functions $\varphi, \psi \in$ D* the so-called *Lagrange identity*

$$l[\varphi]\bar{\psi} - \varphi l[\bar{\psi}] = \frac{d}{dt}[\varphi, \psi]_t,$$

where $[\varphi, \psi]_t$ is the bilinear form

$$[\varphi, \psi]_t = \sum_{k=1}^{n} \{\varphi^{[k-1]}(t)\overline{\psi^{[2n-k]}(t)} - \varphi^{[2n-k]}(t)\overline{\psi^{[k-1]}(t)}\}.$$

We shall use the Lagrange identity in the form

$$\int_\alpha^\beta l[\varphi(t)]\overline{\psi(t)}\,dt - \int_\alpha^\beta \varphi(t)\overline{l[\psi(t)]}\,dt = [\varphi, \psi]_\alpha^\beta,$$

where $[\alpha, \beta]$ is an arbitrary closed subinterval of (a, b) and

$$[\varphi, \psi]_\alpha^\beta = [\varphi, \psi]_\beta - [\varphi, \psi]_\alpha.$$

Since each of the integrals

$$\int_\alpha^\beta l[\varphi(t)]\,\overline{\psi(t)}\,dt, \quad \int_\alpha^\beta \varphi(t)\,l[\overline{\psi(t)}]\,dt$$

exists for $\alpha = a$ and $\beta = b$, the bilinear form $[\varphi, \psi]_t$ has a finite value at the end points of the interval (a, b), whether these end points are regular or singular; for the value of the bilinear form at a singular point we take the limit $[\varphi, \psi]_t$ as t tends to this point.

Let l be a quasi-differential operation, regular or singular in the interval (a, b), and let $g(t)$ be a complex-valued measurable function defined in this interval. It is natural to define a *solution* of the quasi-differential equation

$$l[y] - \lambda y = g(t) \tag{3}$$

for a given value of the parameter λ as any function $\varphi(t)$ such that $\varphi(t)$ and its quasi-derivatives up to the $(2n-1)$th order are absolutely continuous and such that this equation is satisfied almost everywhere in (a, b).

The following existence theorem holds for quasi-differential equations and can be proved easily by using *Picard's method* of successive approximations.

THEOREM: *The quasi-differential equation* (3) *has a solution. Moreover, it has only one solution which satisfies the Cauchy conditions*

$$\varphi^{[k]}(t_0) = \alpha_k \quad (k = 0, 1, 2, \ldots, 2n-1),$$

where t_0 is an arbitrary interior point of the interval (a, b) or a regular end point.

We note that if the operation l is regular in the interval (a, b), then the solution of equation (3) belongs to $L^2(a, b)$. If, in addition, $g(t) \in L^2(a, b)$ then, by equation (3) itself, this solution belongs to D^*.

From the existence theorem it follows, in particular, that the linear manifold of solutions of the homogeneous equation

$$l[u] - \lambda u = 0 \tag{4}$$

has dimension $2n$. Solutions $u_1, u_2, \ldots, u_{2n}$ of equation (4) are linearly independent if and only if the Wronskian determinant

$$w[u_1, u_2, \ldots, u_{2n}] = \operatorname{Det}(u_i^{[k-1]}(t))$$

does not vanish. An arbitrary system of $2n$ linearly independent solutions of equation (4) is called *fundamental.*

If the functions

$$(5) \qquad u_1(t), u_2(t), \ldots, u_{2n}(t)$$

form a fundamental system of solutions of the equation (4), then any solution of the non-homogeneous equation (3) can be represented in the form

$$\varphi(t) = \sum_{k=1}^{2n} u_k(t) \int_{t_0}^{t} v_k(s) g(s)\, ds + \sum_{k=1}^{2n} c_k u_k(t).$$

This result can be easily verified if one modifies properly the classical method of variation of parameters. Here, as in the classical case, it is established that the functions

$$(6) \qquad v_1(t), v_2(t), \ldots, v_{2n}(t)$$

form a fundamental system of solutions of equation (4). It is called the *adjoint* system of the system (5).

For what follows we shall need the following simple

LEMMA: *If the operation l is regular in the interval (a, b), then equation* (3) *under the boundary conditions*

$$y^{[k]}(a) = y^{[k]}(b) = 0 \quad (k = 0, 1, 2, \ldots, 2n-1)$$

has a solution if and only if the right member $g(t)$ is orthogonal to the $2n$-dimensional manifold of solutions of the homogeneous equation (4).

Proof: Let $\varphi(t)$ be the solution of (3) which satisfies the conditions

$$\varphi^{[k]}(b) = 0 \quad (k = 0, 1, 2, \ldots, 2n-1).$$

Applying the Lagrange identity to the function $\varphi(t)$ and some function $u_i(t)$ of a fundamental system for equation (4), we obtain

$$(7) \qquad \sum_{k=1}^{n} \{u_i^{[k-1]}(a)\, \overline{\varphi^{[2n-k]}(a)} - u_i^{[2n-k]}(a)\, \overline{\varphi^{[k-1]}(a)}\} = \int_a^b u_i(s) \overline{g(s)}\, ds.$$

Imposing the fundamental system to the initial conditions

$$u_i^{[k-1]}(a) = \begin{cases} 0 & (i \neq k), \\ 1 & (i = k), \end{cases}$$

we obtain

$$\text{(8)}\qquad \overline{\varphi^{[r]}(a)} = \begin{cases} -\displaystyle\int_a^b u_{2n-r}(s)\overline{g(s)}\,ds & (r = 0, 1, \ldots, n-1), \\ \displaystyle\int_a^b u_{2n-r}(s)\overline{g(s)}\,ds & (r = n, n+1, \ldots, 2n-1), \end{cases}$$

which yields the assertion of the lemma.

2. Regular Differential Operators

Let l be a regular quasi-differential operation defined on the interval (a, b). If φ and ψ are arbitrary functions of D*, then the difference

$$(L^*\varphi, \psi) - (\varphi, L^*\psi) = [\varphi, \psi]_a^b$$

in general is not equal to zero and hence L^* is not a symmetric operator. In order that the right member of the above relation vanish, it is necessary to impose an additional condition on φ and ψ and, moreover, to replace the domain D by a subset of D. In any case it is sufficient to require that each of the functions φ and ψ satisfy the relations

$$\text{(1)}\qquad y^{[k]}(a) = y^{[k]}(b) = 0 \quad (k = 0, 1, \ldots, 2n-1).$$

We denote by D the set of all functions in D* which satisfy the $4n$ conditions (1).

It is natural to expect that the operator L, with domain $D_L = D$, and which coincides with L^* in this domain, is a symmetric operator. Since

$$(L\varphi, \psi) = (\varphi, L\psi)$$

for all functions $\varphi, \psi \in D$, it is necessary only to establish that the manifold D is dense in $L^2(a, b)$. This fact is obvious if l is a differential operation; since in that case the manifold D contains, for example, all polynomials.

For the proof that D is dense in $L^2(a, b)$ in the case of a quasi-differential operation l, we note that, by the lemma of Section 1, one can decompose $L^2(a, b)$ into an orthogonal sum

$$\text{(2)}\qquad L^2(a, b) = \Delta_L \oplus \mathfrak{N}_0,$$

where $\mathfrak{N}_0$ is the $2n$-dimensional manifold of solutions of the homogeneous equation

$$l[u] = 0.$$

Let us assume that $(h, \varphi) = 0$ for all $\varphi \in \mathrm{D}$. We must show that $h = 0$. With this aim, we denote by ψ an arbitrary solution of the equation

$$l\,[\psi] = h.$$

By the Lagrange identity,

$$(\psi, l[\varphi]) = (l[\psi], \varphi) = (h, \varphi) = 0.$$

Since $l[\varphi] = L^*\varphi = L\varphi \in \Delta_L$, then by (2) $\varphi \in \mathfrak{N}_0$, i.e.,

$$l[\psi] = 0.$$

Hence, $h = 0$, which was to be proved.

We now prove that the operator L^* is adjoint to L, which justifies the relation used. For the moment we denote the operator adjoint to L by M. Let $\psi \in \mathrm{D}_M$ and let

$$M\,\psi = \chi.$$

Let ψ_0 denote any solution of the equation

$$l[y] = \chi.$$

By the Lagrange identity,

$$(\varphi, \chi) = (\varphi, l[\psi_0]) = (l[\varphi], \psi_0) = (L\varphi, \psi_0).$$

for each $\varphi \in \mathrm{D}$. On the other hand, by the definition of the adjoint operator,

$$(\varphi, \chi) = (\varphi, M\psi) = (L\varphi, \psi),$$

and, hence,

$$(L\varphi, \psi - \psi_0) = 0.$$

Now, since $\varphi \in \mathrm{D}_L$ is arbitrary, it follows from the decomposition (2) that

$$\psi - \psi_0 \in \mathfrak{N}_0.$$

This relation shows that $\psi \in \mathrm{D}^*$ and, therefore, that

$$M\psi = \chi = l[\psi_0] = l[\psi].$$

And so we have also shown that

$$M \subset L^*.$$

On the other hand, for arbitrary $\varphi \in \mathrm{D}$ and $\psi \in \mathrm{D}^*$,

$$(L\varphi, \psi) - (\varphi, L^*\psi) = [\varphi, \psi]_a^b = 0.$$

Hence,

$$L^* \subset M,$$

which implies that

$$M = L^*.$$

Hence, our assertion is proved.

The reader can easily verify that L^{**} coincides with L, whence it follows that the operator L is closed.

Since the equation

$$L^*\psi - \lambda\psi = 0$$

has (for each λ) $2n$ linearly independent solutions, the deficiency indices of the operator L are $(2n, 2n)$. Every symmetric operator generated by the operation l is an extension of L. Therefore, it is natural to call *L a regular quasi-differential operator of order $2n$ with minimal domain.*

3. Self-Adjoint Extensions[1] of a Regular Differential Operator

To describe all self-adjoint extensions of an operator L, it is sufficient to indicate the domains of all these extensions. We show that the domain of each self-adjoint extension of the operator L can be defined with the aid of certain boundary conditions, and we characterize all such conditions.

Let $\tilde{L}$ denote some self-adjoint extension of L. In order that a function φ in D_{L^*} also belong to $D_{\tilde{L}}$ it is necessary and sufficient that

$$(L^*\varphi, \psi) = (\varphi, \tilde{L}\psi),$$

or, by the Lagrange identity, that

$$(1) \qquad [\varphi, \psi]_a^b = 0$$

for all $\psi \in D_{\tilde{L}}$. Since

$$\dim D_{\tilde{L}} = 2n \qquad (\text{mod } D_L),$$

there exists $2n$ functions $w_1, w_2, \ldots, w_{2n}$ in $D_{\tilde{L}}$ such that each function $\psi \in D_{\tilde{L}}$ can be represented in the form

$$\psi = \psi_0 + \sum_{k=1}^{2n} \alpha_k w_k \qquad (\psi_0 \in D_L).$$

By condition (1) of section (2),

$$[\varphi, \psi_0]_a^b = 0$$

for each function φ in D_{L^*}. Therefore, condition (1), which contains an "arbitrary" function ψ, is equivalent to the set of $2n$ conditions

$$(2) \qquad [\varphi, w_k]_a^b = 0 \qquad (k = 1, 2, \ldots, 2n),$$

---

[1] In the present and following sections, self-adjoint extensions are always of the first kind. For the general form of boundary conditions which characterize self-adjoint extensions of differential operators cf. M. G. Krein [4], Vol. I, and also A. A. Graff [1], Vol. I.

where the functions $w_1, w_2, \ldots, w_{2n}$ belong to $\mathrm{D}_{\tilde{L}}$ and satisfy the conditions

$$(3) \qquad [w_i, w_k]_a^b = 0 \quad (i, k = 1, 2, \ldots, 2n).$$

Equation (2) can be considered as a system of boundary conditions and equation (3) as a property of the coefficients of these conditions. The boundary conditions (2) can be written in the form of $2n$ linearly independent equations

$$(2') \qquad \sum_{k=1}^{2n} \alpha_{ik}\varphi^{[k-1]}(a) + \sum_{k=1}^{2n} \beta_{ik}\varphi^{[k-1]}(b) = 0 \quad (i = 1, 2, \ldots, 2n),$$

where

$$(4) \qquad \begin{aligned} \alpha_{ik} &= \overline{w_i^{[2n-k]}(a)}, \; \alpha_{i,\,n+k} = -\overline{w_i^{[n-k]}(a)}, \\ \beta_{ik} &= -\overline{w_i^{[2n-k]}(b)}, \; \beta_{i,\,n+k} = \overline{w_i^{[n-k]}(b)}, \end{aligned} \qquad (k = 1, 2, \ldots, n).$$

Then equation (3) takes the form

$$(3') \qquad \begin{aligned} &\sum_{r=1}^{n} \alpha_{i,r}\,\bar{\alpha}_{k,\,2n-r+1} - \sum_{r=1}^{n} \alpha_{i,\,2n-r+1}\,\bar{\alpha}_{k,\,r} = \\ &= \sum_{r=1}^{n} \beta_{i,\,r}\bar{\beta}_{k,\,2n-r+1} - \sum_{r=1}^{n} \beta_{i,\,2n-r+1}\bar{\beta}_{k,\,r} \end{aligned} \qquad (i, k = 1, 2, \ldots, 2n).$$

Thus, the domain of each self-adjoint extension $\tilde{L}$ of L consists of all the functions $\varphi \in \mathrm{D}_{L^*}$ which satisfy the $2n$ boundary conditions of the form (2′) with coefficients having the properties (3′).

We assume now a system of $2n$ boundary conditions of the form (2′) with coefficients which satisfy conditions (3′) and we prove that such a system determines in the sense mentioned above the domain of a self-adjoint extension of the operator L.

Let us assume for the moment that there exists $2n$ functions $w_1, w_2, \ldots, w_{2n}$ in D_{L^*} which satisfy conditions (4). Then condition (2′) and equation (3′) take the forms (2) and (3), respectively. We must show that condition (2), with the functions $w_k \in \mathrm{D}_{L^*}$ which satisfy relations (3), determine a self-adjoint extension of L. With this aim we denote by $\tilde{\mathrm{D}}$ the set of all functions $\varphi \in \mathrm{D}_{L^*}$ which satisfy conditions (2) and by D' the set of all functions of the form

$$\psi = \psi_0 + \sum_{k=1}^{2n} \alpha_k w_k,$$

where $\psi_0 \in \mathrm{D}_L$ and $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ are arbitrary constants. Condition (2) is equivalent to

$$(5) \qquad [\varphi, \psi]_a^b = 0$$

for every $\psi \in \mathrm{D}'$. Since every function in D' satisfies this condition, relation

(3) holds. Hence, $\mathrm{D}' \subset \mathrm{D}$. But since both manifolds $\tilde{\mathrm{D}}$ and D' are $2n$-dimensional modulo D_L, $\tilde{\mathrm{D}} = \mathrm{D}'$. Therefore, $\tilde{\mathrm{D}}$ ($\mathrm{D}_L \subset \tilde{\mathrm{D}} \subset \mathrm{D}_{L^*}$) is the set of all functions $\varphi \in \mathrm{D}_{L^*}$ for which equation (5) is satisfied for all $\psi \in \tilde{\mathrm{D}}$. By the lemma in Section 41, the operator $\tilde{L} \subset L^*$ with domain $\tilde{\mathrm{D}}$ is a self-adjoint extension of L.

It remains to show that there exist functions in D_{L^*} which, together with their quasi-derivatives up to and including the $(2n-1)$th order, assume prescribed values at both end points of the interval $[a, b]$. Let $g_1(t)$ be any function which is orthogonal to the manifold of solutions of the homogeneous equation (5) of Section 1. Let $\omega_1(t)$ denote the solution of the equation

$$l[y] - \lambda y = g_1(t),$$

which satisfies the given conditions at the point a. By formula (8) of Section 1, $\omega^{k}(b) = 0$ $(k = 0, 1, 2, \ldots, 2n-1)$. We construct an analogous function $\omega_2(t)$ which satisfies the given conditions at the right end point. Then the sum $\omega_1(t) + \omega_2(t)$ is a function in D_{L^*} which satisfies all of the required conditions.

Thus, we have proved the following theorem.

THEOREM: *A system of linearly independent boundary conditions* (2′) *determines a self-adjoint extension of L if and only if there are* $2n$ boundary *conditions and the coefficients in* (2′) *have property* (3′).

By means of well-known calculations, it is easy to construct the Green's function $\tilde{G}(t, s)$ of an arbitrary self-adjoint extension $\tilde{L}$ of L. The function $\tilde{G}(t, s)$ is a Hilbert-Schmidt kernel and the integral operator $\tilde{K}$ determined by this kernel is related to the operator $\tilde{L}$ by the formula

$$\tilde{L} = \tilde{K}^{-1}.$$

It follows that the spectrum of the operator $\tilde{L}$ consists only of eigenvalues with the unique limit point infinity.

In conclusion, we note that if one omits equation (3′) then the boundary conditions (2′) determine a quasi-self-adjoint extension of the operator L in the sense of Section 5 of Appendix I.

4. Singular Differential Operators[2]

We begin with the case in which the interval is $(0, \infty)$, i.e., the right end point of the interval is singular. We assume the left end point to be regular.

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[2] Sections 4, 5, and 6 of the present appendix are excerpts of the dissertation of I. M. Glazman [1], [3], Vol. I.

Let us define the operator L' by the equation

$$L'\varphi = l[\varphi]$$

on the manifold $\mathrm{D}_{L'}$ of all functions in D^* which vanish outside a finite interval and which satisfy the conditions

$$(1) \qquad \varphi(0) = \varphi^{[1]}(0) = \ldots = \varphi^{[2n-1]}(0) = 0.$$

Since the operator L' is obviously symmetric, it has a closure, which we denote by L and call the singular quasi-differential operator, with *minimal domain* generated by the operation l. The following theorem justifies this designation.

THEOREM 1: *The operator L^* is adjoint to the operator L.*

Proof: As in Section 2, we denote by M the operator adjoint to L. The relation

$$\mathrm{D}_{L^*} \subset \mathrm{D}_M$$

and the equation

$$M\psi = l[\psi]$$

for $\psi \in \mathrm{D}_{L^*}$ are obvious. Hence, it remains only to prove that

$$\mathrm{D}_M \subset \mathrm{D}_{L^*}.$$

With this aim, as in Section 2, we choose a function $\psi \in \mathrm{D}_M$ and let

$$M\psi = \kappa.$$

Furthermore, let ψ_0 denote some solution of the equation

$$l[y] = \chi.$$

Now it remains to prove that

$$\psi(t) = \psi_0(t) + u(t),$$

where $u(t)$ is some solution of the equation

$$(2) \qquad l[y] = 0.$$

Now we choose a function $g(t) \in L^2(0, \infty)$ which vanishes for $t \geqq a$ and is orthogonal to the $2n$ dimensional manifold of solutions of equation (2) in the interval $0 \leqq t \leqq a$. Then, by Section 1, there exists a function $\varphi(t)$ which equals zero for $t \geqq a$ and satisfies the equation

$$l[\varphi] = g$$

and also condition (1). Obviously, $\varphi \in \mathrm{D}_{L'} \subset \mathrm{D}_L$ and, hence,

$$(g, \psi) = (L\varphi, \psi) = (\varphi, M\psi) = (\varphi, \chi).$$

But, on the other hand,

$$(g, \psi_0) = (l[\varphi], \psi_0) = (\varphi, l[\psi_0]) = (\varphi, \chi).$$

Therefore,

$$(g, \psi - \psi_0) = 0.$$

Since $g(t)$ is arbitrary,

$$\psi(t) - \psi_0(t) = u(t),$$

where $u(t)$ is a solution of equation (2).

Remark: Each function in D_L satisfies condition (1). Indeed, let $\psi(t)$ be a function in D_{L^*} which vanishes for $t \geqq a$. Then, for $\varphi \in D_L$, the equations

$$(L\varphi, \psi) = (\varphi, L^*\psi),$$
$$(l[\varphi], \psi) = [\varphi, \psi]_0^a + (\varphi, l[\psi])$$

imply that

$$[\varphi, \psi]_0^a = 0$$

and, hence,

$$[\varphi, \psi]_0 = 0.$$

Since the values $\psi^{k-1}(0)$ $(k = 1, 2, \ldots, 2n)$ are arbitrary, this means that the function $\varphi(t)$ satisfies condition (1).

We come now to the question about deficiency numbers of the operator L. They are identical since the coefficients of the operator l are real.

THEOREM 2: *The deficiency number m of a quasi-differential operator of order $2n$ on an interval with a singular end point satisfies the inequality*

$$n \leqq m \leqq 2n. \tag{3}$$

Proof: The right inequality follows immediately from Theorem 1. For the proof of the left inequality we use the first Neumann formula (Section 80)

$$D_{L^*} = D_L \oplus \mathfrak{N}_\lambda \oplus \mathfrak{N}_{\bar{\lambda}}.$$

By the equality of the dimensions of the subspaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\bar{\lambda}}$ it is sufficient to prove that

$$\dim D_{L^*} \geqq 2n \quad (\mathrm{mod}\ D_L).$$

With this aim, we prove the existence of $2n$ linearly independent functions which, together with their linear envelope, lie in D_{L^*} but (with the exception of the function zero) outside of D_L. It is not difficult to see that it is possible to take as such functions any linearly independent functions $\psi_1(t)$, $\psi_2(t)$, $\ldots$, $\psi_{2n}(t)$ from D_{L^*} which satisfy the condition

$$\mathrm{Det}\,(\psi_i^{[k-1]}(0)) \neq 0.$$

Then from the assumption

$$\psi = \sum_{i=1}^{2n} \alpha_i \psi_i \in \mathrm{D}_L$$

follows (cf. the remark to Theorem 1) the equation

$$\psi(0) = \psi^{[1]}(0) = \ldots = \psi^{[2n-1]}(0) = 0,$$

and, hence

$$\psi(t) = 0.$$

From Theorems 1 and 2 it follows that the number of solutions of the quasi-differential equation

$$l[u] - \lambda u = 0 \qquad (\Im\lambda \neq 0),$$

which belong to $L^2(0, \infty)$ does not depend on λ and is not less than half of the order of the operation l.

The integer m in inequality (3) can be chosen arbitrarily between n and $2n$. It is interesting to note that for $l_1 = -i(1+t)\,\mathrm{D}\,(1+t)$ and $l_2 = \mathrm{D}^2 - 1$ the operation

$$l = l_1^{m-n} l_2^{2n-m} l_1^{m-n} \quad (n \leqq m \leqq 2n)$$

generates an operator of order $2n$, on the interval $(0, \infty)$, with minimal domain and with deficiency indices (m, m). The verification of this fact is left to the reader.[3]

The investigations relating to the case of the interval $(-\infty, \infty)$ with two singular end points is similar to the one given above.

The operator L with minimal domain in this case is defined as the closure of the operator L', but it is necessary to omit condition (1). Theorem 1, which characterizes the operator L^*, is preserved. Theorem 2 is replaced by the following theorem.

THEOREM 3: *Let $L^{(-)}$ and $L^{(+)}$ be two singular quasi-differential operators with minimal domains, generated by the operator l on the intervals $(-\infty, 0)$ and $(0, \infty)$, respectively. Under these conditions the deficiency number m of the operator L is defined by the formula*

$$m = m^{(-)} + m^{(+)} - 2n,$$

where $m^{(-)}$ and $m^{(+)}$ are the deficiency numbers of the operators $L^{(-)}$ and $L^{(+)}$ respectively.

Proof: Let D_0 denote the linear manifold of functions $\varphi \in \mathrm{D}_{L'}$ such that

$$\varphi(0) = \varphi^{[1]}(0) = \ldots = \varphi^{[2n-1]}(0) = 0,$$

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[3] In this connection cf. I. Glazman [1], Vol. I.

and let L_0' denote the restriction of the operator L' to D_0. Let L_0 be the closure of L_0'. Obviously

$$\dim \mathrm{D}_L = 2n \quad (\mathrm{mod}\ \mathrm{D}_{L_0}).$$

But, by a remark in Section 80 (page 100), $\dim \mathrm{D}_L$ (mod D_{L_0}) equals the difference of the deficiency numbers of L_0 and L. Hence, the deficiency number of the operator L_0 equals

$$m_0 = m + 2n.$$

On the other hand, the operator L_0 is reduced by the subspaces $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Hence,

$$L_0 = L^{(-)} \oplus L^{(+)},$$

so that

$$m_0 = m^{(-)} + m^{(+)}$$

and

$$m = m^{(-)} + m^{(+)} - 2n,$$

which was to be proved.

5. Self-Adjoint Extensions of a Singular Differential Operator

Self-adjoint extensions of a singular operator, analogous to self-adjoint extensions of a regular operator, can be characterized by means of a system of boundary conditions that have a structure more complicated than the corresponding conditions of Section 3. Let us consider the case when only one end point of the interval is singular.

The general theory of extensions (cf. Section 80) yields

THEOREM 1: *Let the deficiency indices of the operator L be* (m, m) $(n \leqq m \leqq 2n)$ *and let the functions*

$$u_k(t; \lambda) \in L^2(0, \infty) \quad (k = 1, 2, \ldots, m)$$

form an orthonormal system of solutions of the equation

$$l[u] - \lambda u = 0$$

for some fixed non-real value λ. *Under these conditions there exists a one-to-one correspondence between the class of all self-adjoint extensions* L_θ *of the operator L and the class of all unitary matrices* $\theta = (\theta_{ik})$ *of order m. This correspondence is defined by the formula*

$$\mathrm{D}_{L\theta} = \mathrm{D}_L \oplus \Gamma_\theta, \tag{1}$$

where Γ_θ *is the linear envelope of the functions*

$$w_i^{(\theta)}(t; \lambda) = u_i(t; \lambda) + \sum_{k=1}^{m} \theta_{ik} \overline{u_k(t;\lambda)}.$$

Proof: It is sufficient to note that the functions $u_k(t; \lambda)$ form an orthonormal system of solutions of the equation

$$L^*g - \bar{\lambda}g = 0.$$

We now concern ourselves with the clarification of the boundary conditions which characterize the functions in D_{L_θ}.

THEOREM 2: *Let L_θ be a self-adjoint extension of the operator L with deficiency indices (m, m), defined in the sense of Theorem 1 by a unitary matrix θ. Furthermore, let the functions $w_i^{(\theta)}(t; \lambda)$ $(i = 1, 2, \ldots, m)$ have the same meaning as in Theorem 1. Then the domain D_{L_θ} of the operator L_θ consists of all the functions $\varphi(t)$ in D_{L^*} which satisfy the m conditions*

$$[\varphi, w_i^{(\theta)}]_0^\infty = 0 \quad (i = 1, 2, \ldots, m). \tag{2}$$

Proof: By the Lagrange identity, the functions $\varphi(t) \in D_{L^*}$ belong to D_{L_θ} if and only if the condition

$$[\varphi, \psi]_0^\infty = 0 \tag{3}$$

is satisfied for all $\psi(t)$ in D_{L_θ}. It remains to show that the system of conditions (3), which contain the arbitrary functions $\psi(t) \in D_{L^*}$ is reduced for $\varphi(t) \in D_{L^*}$ to a system of m conditions (2).

By formula (1), the function $\psi(t)$ has the representation

$$\psi(t) = \psi_0(t) + \sum_{i=1}^{m} c_i w_i^{(\theta)}(t; \lambda) \qquad (\varphi_0(t) \in D_L).$$

Therefore,

$$[\varphi, \psi]_0^\infty = [\varphi, \varphi_0]_0^\infty + \sum_{i=1}^{m} c_i [\varphi, w_i^{(\theta)}]_0^\infty.$$

Since $\varphi_0(t) \in D_L$,

$$[\varphi, \varphi_0]_0^\infty = 0$$

for $\varphi(t) \in D_{L^*}$. Since the constants $c_i (i = 1, 2, \ldots, m)$ are arbitrary, conditions (3) and (2) are equivalent.

We call equations (2), which involve passage to a limit, boundary conditions. The following theorem gives a case for which the boundary conditions involve no passage to a limit.

THEOREM 3: *If the deficiency indices of the operator L are (n, n), then the system of boundary conditions which define the self-adjoint extension L_θ of the operator L has the form*

$$[\varphi, w_i^{(\theta)}]_0 = 0 \quad (i = 1, 2, \ldots, n). \tag{4}$$

Proof: It suffices to show that, under the conditions of the theorem,

$$[\varphi, \psi]_\infty = 0$$

for all $\varphi(t), \psi(t) \in D_{L^*}$. With this aim, we choose in D_{L^*} the functions

$$\psi_i(t) \quad (i = 1, 2, \ldots, 2n), \quad \psi_i(t) = 0 \text{ for } t > a > 0,$$

so that

$$\operatorname{Det} (\psi_i^{[k-1]}(0)) \neq 0.$$

The linear envelope of the functions $\psi_i(t)$ and D_L have only the zero elements in common. On the other hand, by the assumption on the deficiency indices of L,

$$\dim D_{L^*} = 2n \pmod{D_L}.$$

Hence, for each function $\varphi(t) \in D_{L^*}$ there exists constants $a_1, a_2, \ldots, a_{2n}$ such that

$$\varphi(t) - \sum_{i=1}^{2n} a_i\psi_i(t) = \varphi_0(t) \in D_L.$$

Therefore, since the functions $\psi_i(t)$ $(i = 1, 2, \ldots, 2n)$ vanish for $t > a$ and $\varphi_0^{k-1}(0) = 0$ $(k = 1, 2, \ldots, 2n)$,

$$[\varphi, \psi]_\infty = [\varphi_0 + \sum_{i=1}^{2n} a_i \psi_i, \psi]_\infty = [\varphi_0, \psi]_\infty = [\varphi_0, \psi]_0^\infty = 0.$$

We must note here that, in the general case, the boundary condition (2) depends essentially on the functions $w_i^{(\theta)}(i=1, 2, \ldots, m)$, so that it is not defined until the operator L is given. However, if the deficiency indices of the operator L are (n, n), then the boundary condition (4) can be freed from dependence on the functions $w_i^{(\theta)}(t; \lambda)$ by the method of Section 3 and then (4) does not depend on L.

In the case of the deficiency indices $(2n, 2n)$ the number of boundary conditions which determine a self-adjoint extension of a singular operator is equal to $2n$ as in the case of a regular operator. In view of this circumstance and some other considerations (cf. Theorem 2 of the following section) we call a singular operator with the deficiency indices $(2n, 2n)$ *quasi-regular*.

For the differential operator of the second order, the only possible cases are $m=2$ and $m=1$. This was first observed by Weyl who called $m=2$ the case of the *limit circle*, and $m=1$, the case of the *limit point*. We shall clarify the meaning of this terminology in Section 9.

Since the construction of the boundary conditions for the case of an interval with two singular end points can be reduced without difficulty to the method used in the present section, it is omitted.

In conclusion, we consider briefly the question of self-adjoint extensions L_0 which are real with respect to the operator of complex-conjugation in $L^2(0, \infty)$ (cf. Section 45). The operator L^* is real with respect to the operator of complex conjugation, since the coefficients which generate the quasi-differential operators are real. So L_θ is real if and only if the manifold D_{L_θ} contains the conjugate $\overline{\varphi(t)}$ of each function $\varphi(t)$ in D_{L_θ}. By means of this fact and formula (1), it is easy to establish that the operator L_θ is real if and only if the unitary matrix $\theta = (\theta_{ik})$ is symmetric,[4] i.e.,

$$\theta_{ik} = \theta_{ki} \qquad (i, k = 1, 2, \ldots, m).$$

In the following section we shall need the existence of only one real extension. This follows from formula (1) if we choose for θ the unit matrix.

6. The Resolvents of Self-Adjoint Extensions

In the present section we show that the resolvents of self-adjoint extensions of a quasi-differential operator L are integral operators and we characterize the class of the kernels of these integral operators for distinct deficiency indices of the operator L. We begin with two lemmas.

LEMMA 1: *The resolvent of a self-adjoint operator which is defined in* $L^2(0, \infty)$ *and real with respect to the operation of complex conjugation on all functions* $g(t) \in L^2(0, \infty)$ *which vanish outside of a finite interval, has the form*

$$R_\lambda g = \int_0^\infty K(t, s; \lambda)\, g(s)\, ds.$$

The kernel $K(t, s; \lambda)$ *is continuous in* s *and* t *for*

$$t > 0, \; s > 0 \qquad (t \neq s),$$

and

$$K(t, s; \lambda) = K(s, t; \lambda).$$

The proof follows immediately from the relation $R_\lambda^* = IR_\lambda I$ (cf. Section 45) and the fact that the function $g(t)$ is arbitrary.

LEMMA 2: *Let* $\mathbf{L}$ *be a homogeneous and additive (but not necessarily bounded) functional defined for all functions* $g(t)$ *of* $L^2(0, \infty)$ *which vanish outside of a finite interval. Suppose that* $\mathbf{L}$ *is bounded on* $L^2(0, a)$ *for each finite* $a > 0$. *Then* $\mathbf{L}$ *has the representation*

$$\mathbf{L}(g) = \int_0^\infty g(s)\, \overline{h(s)}\, ds. \tag{1}$$

Here $h(t)$ *is a function determined by the functional* $\mathbf{L}$ *and belongs to* $L^2(0, a)$ *for each finite* $a > 0$.

---

[4] In particular, it follows that in the case with the deficiency indices (1, 1) every self-adjoint extension is real.

The proof follows from the theorem of Riesz on the representation of a linear functional (cf. Section 16), according to which

$$\text{(2)} \qquad \mathbf{L}(g) = \int_0^a g(s)\overline{h_a(s)}\,ds \qquad (h_a(s) \in L^2(0, a))$$

for each $g(t) \in L^2(0, a)$. The functions $h_a(t)$ which appears in this representation have the following property. If $a_1 < a_2$ then

$$h_{a_1}(t) = h_{a_2}(t)$$

for almost all $t < a_1$. This follows from the fact that for each function $g(t)$ in $L^2(0, a_1)$, there are the two representations

$$\mathbf{L}(g) = \int_0^{a_1} g(s)\overline{h_{a_1}(s)}\,ds,$$

$$\mathbf{L}(g) = \int_0^{a_2} g(s)\overline{h_{a_2}(s)}\,ds = \int_0^{a_1} g(s)\overline{h_{a_2}(s)}\,ds.$$

Hence,

$$\int_0^{a_1} [h_{a_2}(s) - h_{a_1}(s)]\overline{g(s)}\,ds = 0.$$

The fact just proved permits us to write formula (2) in the form (1).

THEOREM 1: *The resolvent $\tilde{R}_\lambda$ of each self-adjoint extension $\tilde{L}$ of a quasi-differential operator L with deficiency indices (m, m) at the points λ of regular type, is an integral operator.*

Proof: First we give the proof for an arbitrary real self-adjoint extension, after which we show that if the theorem is valid for one of the self-adjoint extensions, then it is valid for all self-adjoint extensions. Let $\tilde{L}_0$ be a real self-adjoint extension of L and let $\tilde{R}^0_\lambda$ be the corresponding resolvent (where λ is a fixed point of regular type of the operator L).

Note that the element $\tilde{R}^0_\lambda g$ (for $g(t) \in L^2(0, \infty)$) must be in the manifold of solutions of the inhomogeneous quasi-differential equation

$$\text{(3)} \qquad l[y] - \lambda y = g.$$

Consider the corresponding homogeneous equation,

$$\text{(4)} \qquad l[u] - \lambda u = 0.$$

We choose a fundamental system of solutions $u_i(t; \lambda)$ $(i = 1, 2, \dots, 2n)$ such that the first m of these functions belong to $L^2(0, \infty)$. (Then the solutions $u_{m+1}(t; \lambda), \dots, u_{2n}(t; \lambda)$ together with their non-trivial linear combinations lie outside of $L^2(0, \infty)$). Here we do not assume as we did in the preceding section, that the solutions $u_i(t; \lambda)$ $(i=1, 2, \ldots, m)$ be orthonormal.

By Section 1 of Appendix II, the general solution of equation (3) has the form

$$\varphi(t) = \sum_{k=1}^{2n} u_k(t; \lambda) \int_0^t v_k(s; \lambda) g(s)\, ds + \sum_{k=1}^{2n} c_k u_k(t; \lambda). \tag{5}$$

Here the system $v_k(t; \lambda)$ $(k=1, 2, \ldots, 2n)$ is the adjoint to the fundamental system $u_k(t; \lambda)$ $(k=1, 2, \ldots, 2n)$ of solutions of equation (4). The constants $c_k (k=1, 2, \ldots, 2n)$ in formula (5) are uniquely determined if we require that $\varphi(t) = \tilde{R}_\lambda^0 g$. We now determine the values of the constants for which this condition is satisfied under the assumption that $g(t)$ vanish outside of a finite interval.

First, we must make sure that $\varphi(t)$ belongs to $L^2(0, \infty)$. For $t \geqq a$, $g(t) = 0$ and, hence,

$$\varphi(t) = \sum_{k=1}^{2n} u_k(t; \lambda) \int_0^a v_k(s; \lambda) g(s)\, ds + \sum_{k=1}^{2n} c_k u_k(t; \lambda).$$

Then $\varphi(t)$ belongs to $L^2(0, \infty)$ if and only if

$$c_k = -\int_0^a v_k(s; \lambda) g(s)\, ds = -\int_0^\infty v_k(s; \lambda) g(s)\, ds \quad (k=m+1, m+2, \ldots, 2n).$$

For this choice of these constants $\tilde{R}_\lambda^0$ has the representation

$$\begin{aligned} \tilde{R}_\lambda^0 g &= \sum_{k=1}^{2n} u_k(t; \lambda) \int_0^t v_k(s; \lambda) g(s)\, ds + \\ &+ \sum_{k=1}^{m} c_k u_k(t; \lambda) - \sum_{k=m+1}^{2n} u_k(t; \lambda) \int_0^\infty v_k(s; \lambda) g(s)\, ds = \\ &= \sum_{k=1}^{m} u_k(t; \lambda) \int_0^t v_k(s; \lambda) g(s)\, ds - \\ &- \sum_{k=m+1}^{2n} u_k(t; \lambda) \int_t^\infty v_k(s; \lambda) g(s)\, ds + \sum_{k=1}^{m} c_k u_k(t; \lambda), \end{aligned}$$

or

$$\tilde{R}_\lambda^0 g = \int_0^\infty K(t, s; \lambda) g(s)\, ds + \sum_{k=1}^{m} c_k\, u_k(t; \lambda), \tag{6}$$

where

$$K(t,s;\lambda) = \begin{cases} \sum_{k=1}^{m} u_k(t;\lambda)\, v_k(s;\lambda) & (s \leqq t) \\ -\sum_{k=m+1}^{2n} u_k(t;\lambda)\, v_k(s;\lambda) & (s > t). \end{cases}$$

In representation (6) $g(t)$ is any function of $L^2(0, \infty)$ which vanishes outside of some finite interval and the $c_k (k = 1, 2, \ldots, m)$ are the constants which correspond to it. From this representation it follows immediately that the constants $c_k (k = 1, 2, \ldots, m)$ are homogeneous and additive functionals defined on all such functions in $L^2(0, \infty)$. We show now that these functionals $c_k = c_k(g)$ $(k = 1, 2, \ldots, m)$ satisfy the conditions of lemma 2, i.e., that they are bounded in $L^2(0, a)$ for each $a < \infty$. We apply to both members of equation (6) the operator P_a of projection on the subspace $L^2(0, a)$ and then form the scalar products with $u_i(t; \lambda)$ $(i = 1, 2, \ldots, m)$ to obtain

$$(7) \qquad (P_a \tilde{R}^0_\lambda g, u_i) = (K_a g, u_i) + \sum_{k=1}^{m} c_k (P_a u_k, u_i) \quad (i = 1, 2, \ldots, m),$$

where

$$K_a g = P_a \int_0^\infty K(t, s; \lambda)\, g(s)\, ds.$$

In the system (7) of m equations with m unknowns c_k $(k = 1, 2, \ldots, m)$, the norms of the operators $P_a \tilde{R}^0_\lambda$ are bounded for all positive $a < \infty$ by a number which does not depend on a, the norms of the integral operators K_a in $L^2(0, \infty)$ with Hilbert-Schmidt kernel are bounded for all $a < \infty$ by a number which depends on a and, finally, the Gram determinant

$$\operatorname{Det}\left((P_a u_k, u_i) \right)$$

is different from zero by the linear independence of the solutions $u_k(t; \lambda)$ $(k = 1, 2, \ldots, m)$. It follows that the $c_k (k = 1, 2, \ldots, m)$ are functionals which are bounded in $L^2(0, a)$ for each $a < \infty$ so that, by Lemma 2, they can be represented in the form

$$c_k(g) = \int_0^\infty g(s)\, \psi_k(s; \lambda)\, ds \qquad (k = 1, 2, \ldots, m),$$

where the functions $\psi_k(s; \lambda)$ $(k = 1, 2, \ldots, m)$ belong to $L^2(0, a)$ for each $a < \infty$. Now the representation (6) assumes the form

$$\tilde{R}^0_\lambda g = \int_0^\infty K(t, s; \lambda)\, g(s)\, ds + \sum_{k=1}^{m} u_k(t; \lambda) \int_0^\infty g(s)\, \psi_k(s; \lambda)\, ds,$$

or

$$\tilde{R}^0_\lambda g = \int_0^\infty K_0(t, s; \lambda)\, g(s)\, ds, \tag{8}$$

where

$$K_0(t,s;\lambda) = \begin{cases} \sum_{k=1}^{m} u_k(t;\lambda)\,[v_k(s;\lambda)+\psi_k(s;\lambda)\,] & (s \leqq t), \\ \sum_{k=1}^{m} u_k(t;\lambda)\,\psi_k(s;\lambda) - \sum_{k=m+1}^{2n} u_k(t;\lambda) v_k(s;\lambda) & (s > t). \end{cases} \tag{9}$$

Formula (8) shows that the resolvent $\tilde{R}^0_\lambda$ is an integral operator on all functions $g(t)$ in $L^2(0, \infty)$ which vanish outside of a finite interval.

From these considerations it still does not follow that $\tilde{R}^0_\lambda$ is an integral operator. By the boundedness of the operator $\tilde{R}^0_\lambda$ it follows from formula (8) only that, for each function in $L^2(0, \infty)$,

$$\tilde{R}^0_\lambda g = \underset{n\to\infty}{\text{l.i.m.}} \int_0^n K_0(t,s;\lambda)\, g(s)\, ds. \tag{10}$$

In order to derive from this the desired representation of the resolvent in the form

$$R^0_\lambda g = \int_0^\infty K_0(t,s;\lambda)\, g(s)\, ds \qquad (g(t) \in L^2(0, \infty)), \tag{11}$$

it is sufficient to prove that the integral

$$\int_0^\infty K_0(t, s; \lambda)\, g(s)\, ds$$

exists for each function $g(t) \in L^2(0, \infty)$. For this, in turn, it is sufficient that

$$\int_0^\infty |K_0(t,s;\lambda)|^2 ds < \infty \qquad (0 \leqq t < \infty), \tag{12}$$

or, equivalently, that the kernel $K_0(t, s; \lambda)$ belongs to $L^2(0, \infty)$ as a function of s for each t.

From formula (9) it follows that the kernel $K_0(t, s; \lambda)$ belongs to $L^2(0, \infty)$ as a function of t for each s since $u_k(t; \lambda) \in L^2(0, \infty)$ for $k = 1, 2, \ldots, m$. By Lemma 1, the function $K_0(t, s; \lambda)$, which corresponds to the resolvent of a real self-adjoint extension, is symmetric with respect to the arguments s and t. Therefore, the integral (12) exists and the representation (11) is established.[5]

To complete the proof of the theorem it remains for us to generalize representation (11) to the case of the resolvent $\tilde{R}_\lambda$ of an arbitrary (not necessarily real) extension $\tilde{L}$ of the operator L.

Since the element $\tilde{R}_\lambda g$ $(g(t) \in L^2(0, \infty))$ belongs to the solution manifold of equation (3), we can represent it in the form

$$\tilde{R}_\lambda g = \tilde{R}_\lambda^0 g + \sum_{k=1}^{2n} c_k u_k, \tag{13}$$

where $\tilde{R}_\lambda^0$ is the resolvent of a real self-adjoint extension. From this representation, in correspondence with the choice of the fundamental system $u_k(t; \lambda)$ $(k=1, 2, \ldots, 2n)$, it follows that for $k \leqq m$ the c_k vanish and for $k > m$ the c_k are linear (i.e., homogeneous, additive, and bounded) functionals defined on $L^2(0, \infty)$ and, hence, have the representation

$$c_k = c_k(g) = \int_0^\infty g(s)\, \chi_k(s; \lambda)\, ds, \tag{14}$$

where the functions $\chi_k(s; \lambda)$ $(k=1, 2, \ldots, m)$ belong to $L^2(0, \infty)$.

Substituting (14) into (13) and defining the kernel $\tilde{K}(t, s; \lambda)$ by the equation

$$\tilde{K}(t, s; \lambda) = K_0(t, s; \lambda) + \sum_{k=1}^{m} u_k(t; \lambda) \chi_k(s; \lambda), \tag{15}$$

we finally obtain the integral representation of the resolvent in the form

$$\tilde{R}_\lambda g = \int_0^\infty \tilde{K}(t, s; \lambda)\, g(s)\, ds \qquad (g(t) \in L^2(0, \infty)).$$

Thus, Theorem 1 is completely proved.

THEOREM 2: *The kernel $\tilde{K}(s, t; \lambda)$ of the resolvent $\tilde{R}_\lambda$ of an arbitrary self-adjoint extension $\tilde{L}$ of the operator L with deficiency indices (m, m) satisfies the conditions*

$$\int_0^\infty |\tilde{K}(t, s; \lambda)|^2 ds < \infty, \quad \int_0^\infty |\tilde{K}(t, s; \lambda)|^2 dt < \infty, \tag{16}$$

---

[5] In the above considerations the symmetry of $K_0(s, t; \lambda)$ is essential in the transition from (10) to (11) in order to avoid the question of the behavior of the functions $v_k(s; \lambda)$ $(k=m+1, m+2, \ldots, 2n)$ and $\psi_k(s; \lambda)$ $(k=1, 2, \ldots, m)$ as $s \to \infty$. Furthermore, the symmetry of the kernel $K_0(t, s; \lambda)$ implies certain properties of these functions at infinity (cf. Theorem 3 page 184).

and if the operator is quasi-regular ($m=2n$) *also satisfies the condition*

$$\int_0^\infty\int_0^\infty |K(t,s;\lambda)|^2\,dsdt < \infty. \tag{17}$$

Proof: Property (16) follows immediately from formula (15), since the function $K_0(t, s; \lambda)$ (in each of the variables s and t) and the functions $\chi_k(s; \lambda)$ $(k = 1, 2, \dots, m)$ belong to $L^2(0, \infty)$.

We come now to property (17). We establish first that for any deficiency indices (m, m) $(n \leqq m \leqq 2n)$ of the operator L, the functions $\psi_k(t; \lambda)$ $(k=1, 2, \dots, m)$ and $v_k(t; \lambda)$ $(k=m+1, m+2, \dots, 2n)$ which appears in formula (9) for $K_0(t, s; \lambda)$ belong to $L^2(0, \infty)$. With this aim we again use the symmetry of the function $K_0(t, s; \lambda)$, which implies that

$$\sum_{k=1}^{m} u_k(s;\lambda)\,\psi_k(t;\lambda) - \sum_{k=m+1}^{2n} u_k(s;\lambda)\,v_k(t;\lambda) = \tag{18}$$

$$= \sum_{k=1}^{m} [v_k(s;\lambda) + \psi_k(s;\lambda)\,]\,u_k(t;\lambda).$$

We choose numbers $s_1, s_2, \dots, s_{2n}$ such that the determinant

$$\operatorname{Det}\,(u_k(s_i;\lambda)\,) \neq 0$$

(this is possible since the functions $u_k(s; \lambda)$ are linearly independent) and we let $s=s_i$ $(i=1, 2, \dots, 2n)$ in equation (18). Solving the resulting system of $2n$ equations in terms of the $2n$ functions $\psi_k(t; \lambda)$ $(k=1, 2, \dots, m)$ and $v_k(t; \lambda)$ $(k=m+1, m+2, \dots, 2n)$ we find that each of these functions is some linear combination of the functions $u_1(t; \lambda), \dots, u_m(t; \lambda)$ and, hence, belongs to $L^2(0, \infty)$.

We now establish property (17). In the case of a quasi-regular operator $(m=2n)$ all summands in the first part of formula (9) belong to the space L^2 of functions of two variables in the quadrant $t>0$, $s>0$ and, hence,

$$\int_0^\infty\int_0^\infty |K_0(t,s;\lambda)|^2 dsdt < \infty. \tag{19}$$

Property (17) follows from (19) by means of (15).

From the proof of the theorem it follows that, in the case of a quasi-regular operator L, the resolvent of an arbitrary self-adjoint extension $\tilde{L}$ of L is defined in terms of a Hilbert-Schmidt kernel and therefore is a completely continuous operator.

From Theorem 2 it follows that the real and imaginary parts of the kernel of the resolvent of any self-adjoint extension of the operator are Carleman kernels.[6]

We note that the arguments in the proof of Theorem 2 yield the following theorem.

THEOREM 3: *Let m denote the maximal number of linearly independent solutions of equation* (4) *which belong to* $L^2(0, \infty)$. *If the first m functions in the fundamental system of solutions of equation* (4) *belong to* $L^2(0, \infty)$, *then the last* $2n$-m *functions in the adjoint fundamental system of this equation also belong to* $L^2(0, \infty)$.

The following theorem is concerned with a case when it is possible to deduce the deficiency indices of the operator L from the number of functions in $L^2(0, \infty)$ which satisfy equation (4) with λ real.

THEOREM 4: *In order that an operator L of order* $2n$ *have deficiency indices* $(2n, 2n)$ *it is necessary that the equation*

$$l[u] - \lambda u = 0$$

have $2n$ *solutions in* $L^2(0, \infty)$ *for each real* λ *and sufficient that this equation have* $2n$ *solutions in* $L^2(0, \infty)$ *for at least one real* λ.

Proof: For the proof of the necessity, we recall that in the case of a quasi-regular operator, the resolvent of its self-adjoint extension $\tilde{L}$ is a completely continuous operator and, hence, the spectrum of $\tilde{L}$ consists only of eigenvalues $\lambda_r (r=1, 2, 3, \ldots)$ with the unique limit point $\lambda = \infty$. It follows that all points of the plane are of regular type with respect to L except perhaps for the points $\lambda_r (r = 1, 2, 3, \ldots)$. If λ_r is not of regular type, then λ_r is an eigenvalue of L. This is impossible, since the equation

$$L\varphi - \lambda_r \varphi = 0$$

is equivalent to

$$l\,[\varphi] - \lambda_r \varphi = 0$$

under the conditions

$$\varphi\,(0) = \varphi^{[1]}(0) = \ldots = \varphi^{[2n-1]}\,(0) = 0$$

and, by the existence theorem of Section 1 (Appendix II), has only the solution $\varphi\,(t) \equiv 0$.

---

[6] A Carleman kernel is a measurable (in general, complex valued) function $K(s, t)\ (-\infty < {}^{s}_{t} < \infty)$ for which:

$$1^\circ \qquad \overline{K(s, t)} = K(t, s)$$

almost everywhere in the (s, t)-plane;

$$2^\circ \qquad \int_{-\infty}^{\infty} |K(s, t)|^2\, dt < \infty$$

almost everywhere on the s-axis.

Thus, in the case of deficiency indices $(2n, 2n)$, the whole λ-plane is the field of regularity of the operator L. By the theorem on the invariance of the deficiency indices (Section 78) every solution of equation (4) with arbitrary λ (real or not real) belongs to $L^2(0, \infty)$.

The sufficiency of the theorem follows immediately from Theorem 4 of Section 83.

It is not difficult to transfer the results obtained in the present section to the case of an interval with two singular end points. We sketch the proof of the fundamental Theorem 1 for this case. Here, as in the case of one singular end point, it is sufficient to consider the resolvent of an arbitrary real self-adjoint extension. Let (cf. the proof of Theorem 3, Section 4, Appendix II) the deficiency numbers of the operators L, $L^{(-)}$ and $L^{(+)}$ be m, $m^{(-)}$ and $m^{(+)}$, respectively. By Theorem 3 of Section 4,

$$\begin{aligned} m^{(-)} &= n + p, \\ && (0 \leqq p \leqq n) \\ m^{(+)} &= n + m - p. \end{aligned}$$

We choose a fundamental system of solutions of equation (4) such that the first m solutions $u_k(t; \lambda)$ belong to $L^2(-\infty, \infty)$. In the $(2n-m)$ dimensional linear envelope of the remaining solutions of this system there exist $n-p-m$ linearly independent solutions in $L^2(-\infty, \infty)$, which we denote by $u_{m+1}(t; \lambda), \ldots, u_{n+p}(t; \lambda)$, and $n-p$ linearly independent solutions in $L^2(0, \infty)$, which we denote by $u_{n+p+1}(t; \lambda), \ldots, u_{2n}(t; \lambda)$. We remark that the functions

$$u_{m+1}(t; \lambda), \ldots, u_{n+p}(t; \lambda), u_{n+p+1}(t; \lambda), \ldots, u_{2n}(t; \lambda)$$

are linearly independent. Otherwise a linear combination of them would belong to $L^2(-\infty, \infty)$, which would imply, contrary to assumption, that the deficiency numbers of the operator L exceed m. With the aid of the chosen fundamental system $u_k(t; \lambda)$ $(k=1, 2, \ldots, 2n)$ each solution of the equation

$$l[y] - \lambda y = g,$$

(where $g(t)$ is a function which vanishes outside of a finite interval) can be represented in the form

$$\varphi(t) = \sum_{k=1}^{2n} u_k(t; \lambda) \int_{-\infty}^{t} v_k(s; \lambda) g(s)\, ds + \sum_{k=1}^{2n} c_k u_k(t; \lambda).$$

For suitable choice of the constants c_k, this function coincides with $\tilde{R}_\lambda^0 g$.

An argument similar to that for the case of one singular end point yields a complete proof of Theorem 1 and Theorem 2 for the case of two singular end points.

7. Inversion Formulas Related to Differential Operators of the Second Order

In Section 69 it was proved that every self-adjoint operator A with a simple spectrum determines an isometric mapping V of the space H onto some space L^2_σ. Under this mapping the operator A goes into the operator of the multiplication by the independent variable. If E_λ is the spectral function and g is any generating element of the operator A, then $\sigma(\lambda) = (E_\lambda g, g)$ and the indicated isometric correspondence is defined by the formulas

$$\Phi(\lambda) = Vf, \tag{1}$$

$$f = V^{-1}\Phi(\lambda) = \int_{-\infty}^{\infty} \Phi(\lambda)\, dE_\lambda g, \tag{2}$$

where the element f and the function $\Phi(\lambda)$ are in H and L^2_σ, respectively. If $f \in \mathrm{D}_A$, then instead of formula (2) the following equation holds:

$$Af = \int_{-\infty}^{\infty} \lambda\Phi(\lambda)\, dE_\lambda g.$$

We call formulas (1) and (2) the *inversion formulas* related to the self-adjoint operator A.

In Example C of Section 77 we obtained as the inversion formulas related to the operator of differentiation on the entire real axis, the mutually inverse Fourier-Plancherel transformations of the space $L^2(-\infty, \infty)$ onto itself

$$\Phi(\lambda) = \underset{N\to\infty}{\mathrm{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t) e^{i\lambda t} dt,$$

$$f(t) = \mathrm{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \Phi(\lambda) e^{-i\lambda t} dt.$$

Here any improper element may be taken as a generating element.

Operators with multiple spectrum also generate inversion formulas which are analogous to formulas (1), (2).

The problem of the present and the following section is to obtain inversion formulas connected with singular *quasi-differential operators* on the semi-axis. This problem was solved first for the operator of second order by Weyl.[7] We begin also by considering operators of the second order but we use a very general and powerful method recently discovered by Krein.[8] This method was called by its author the method of the *direction functionals*. It permits us to obtain inversion formulas for quasi-differential operators of any order (cf. the following section).

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[7] Cf. H. Weyl [2], Vol. 1. The results of Weyl were obtained recently by B. M. Levitan [2], Vol. I, by another method.

Thus, let L be a quasi-differential operator with minimal domain, generated by the operation

$$l = -\frac{d}{dt} p \frac{d}{dt} + q$$

on the interval $(0, \infty)$ with one singular end point. We assume that the deficiency indices of L are $(1, 1)$.[9] As we shall see, in this case the spectrum of each of the self-adjoint extensions of L is simple.

Let $\tilde{L}$ be a self-adjoint extension of the operator L which is determined by the boundary conditions[10]

$$p(t)\, \varphi'(t)\, |_{t=0} = \theta \varphi(0) \qquad (\Im \theta = 0),$$

and let $u_1(t; \lambda)$ and $u_2(t; \lambda)$ be the solutions of the equation

$$l[u] - \lambda u = 0 \qquad (\Im \lambda = 0),$$

which satisfy the initial conditions

$$u_1(0; \lambda) = 1, \quad p(t)\, u_1'(t; \lambda)\, |_{t=0} = 0,$$

$$u_2(0; \lambda) = 0, \quad p(t)\, u_2'(t; \lambda)\, |_{t=0} = 1.$$

We denote the set of all functions $f(t)$ in $L^2(0, \infty)$ which vanish outside of a finite interval by $\mathfrak{D}$ and we define on $\mathfrak{D}$ a homogeneous and additive functional

$$\Phi(f; \lambda) = \int_0^\infty f(t)\, u(t; \lambda)\, dt,$$

where

$$u(t; \lambda) = u_1(t; \lambda) + \theta u_2(t; \lambda).$$

Following M. G. Krein, we call $\Phi(f; \lambda)$ a *direction functional.*

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[8] Cf. M. G. Krein [2], [5], Vol. I; M. S. Lifschitz [1], Vol. I, and A. J. Povzner [1], Vol. I.

[9] If the deficiency indices of the operator L are $(2, 2)$, then each self-adjoint extension of this operator has a discrete spectrum. In this case the question of the structure of the inversion formulas does not arise since the role of formula (2) is played by the expansion of the function into orthogonal eigenfunctions, and the role of formula (1) is played by the expressions for the coefficients in the Fourier expansion of the function.

[10] Thus, there remains to consider only one extension, that which is determined by the condition $\varphi(0) = 0$; it corresponds to $\theta = \infty$. The inversion formula established below is extended to this case by means of a passage to a limit.

The direction functional $\Phi(f; \lambda)$ has the following three properties which are essential for what follows.

1° $\Phi(f; \lambda)$ is an analytic function of λ $(-\infty < \lambda < \infty)$ for each fixed function $f \in \mathfrak{D}$.

2° If for some function $f \in \mathfrak{D}$ and some real λ

$$\Phi(f; \lambda) = 0,$$

then the equation

$$(\tilde{L} - \lambda E)\varphi = f$$

has a solution in the class of functions which vanish outside a finite interval.

3° If f is a function in $D_{\tilde{L}}$ which vanishes outside a finite interval, then for each real λ

$$\Phi(\tilde{L}f; \lambda) = \lambda\Phi(f; \lambda).$$

We prove only properties 2° and 3°. We establish 2° first. Let $f(t)=0$ for $t > a$ and assume that

$$\Phi(f; \lambda_0) = 0. \tag{3}$$

We denote by $\varphi(t)$ $(0 \leqq t < \infty)$ the solution of the equation

$$l[\varphi] - \lambda_0\varphi = f,$$

which satisfies the condition

$$\varphi(a) = p(a)\varphi'(a) = 0,$$

(by the uniqueness theorem $\varphi(t) = 0$ for $t \geqq a$). Using the Lagrange identity, we get

$$\Phi(f; \lambda_0) = \int_0^\infty \{l[\varphi] - \lambda_0\varphi\}u(t; \lambda_0)\,dt = [\varphi, u]_0^\infty = [\varphi, u]_0 =$$

$$= \left[p\left(u\frac{d\varphi}{dt} - \varphi\frac{du}{dt}\right)\right]_{t=0} = p(t)\varphi'(t) - \theta\varphi(t)\,|_{t=0},$$

which in combination with (3) yields

$$p(t)\varphi'(t)\,|_{t=0} = \theta\varphi(0),$$

so that $\varphi(t) \in D_{\tilde{L}}$. Thus, property 2° is established.

The validity of 3° follows from the equations

$$\Phi(\tilde{L}f; \lambda) = \int_0^\infty l[f]\,u(t; \lambda)\,dt = [f, u]_0^\infty + \lambda\int_0^\infty f(t)\,u(t; \lambda)dt = \lambda\Phi(f; \lambda).$$

The direction functional $\Phi(f; \lambda)$ generates a mapping of the linear manifold $\mathfrak{D}$ of functions $f(t)$ onto some linear manifold $\mathfrak{D}'$ of functions

$\Phi(\lambda) = \Phi(f; \lambda)$. If a function $f(t)$ belongs to $D_{\tilde{L}}$ then by 3° this mapping takes $\tilde{L}f$ into the function $\lambda\Phi(\lambda)$. This fact permits us to assume that the spectrum of the operator $\tilde{L}$ is simple and that for a proper choice of the generating function the first of the inversion formulas connected with the operator $\tilde{L}$ will have the form

$$\Phi(\lambda) = \int_0^\infty f(t)\, u(t; \lambda)\, dt$$

for $f(t) \in \mathfrak{D}$. This formula is analogous to the Fourier-Plancherel transform.

We now leave these considerations and proceed to the solution of the problem which was stated above. We shall need two propositions due to Krein.[11]

LEMMA: *For each finite interval Δ_0 of the λ axis there exists a function $\psi(t) \in \mathfrak{D}$ such that the function $\Phi(\lambda) = \Phi(\psi; \lambda)$ does not vanish in the interval Δ_0.*

Proof: Let λ_0 be a fixed point of the interval Δ_0 and let $\chi_0(t)$ be a function in $\mathfrak{D}$ such that

$$\Phi(\chi_0; \lambda_0) \neq 0.$$

If $\Phi(\chi_0; \lambda) \neq 0$ in the whole interval, then for the proof of the lemma it is sufficient to choose $\psi = \chi_0$. If at some point $\lambda_1 \in \Delta_0$

$$\Phi(\chi_0; \lambda_1) = 0,$$

then by property 2° there exists a function $\chi_1 \in D_{\tilde{L}}$ such that

$$\tilde{L}\chi_1 - \lambda_1\chi_1 = \chi_0.$$

Applying to both sides of this equation the direction functional and using property 3° we obtain

$$\lambda\Phi(\chi_1; \lambda) - \lambda_1\Phi(\chi_1; \lambda) = \Phi(\chi_0; \lambda),$$

which yields

$$\Phi(\chi_1; \lambda) = \frac{\Phi(\chi_0; \lambda)}{\lambda - \lambda_1}.$$

Thus, replacing χ_0 by χ_1 we see that the function $\Phi(\chi_1; \lambda)$ does not vanish at $\lambda = \lambda_0$ and at $\lambda = \lambda_1$ it has a zero of multiplicity one less than that of the function $\Phi(\chi_0; \lambda)$. Since the analytic function $\Phi(\chi_0; \lambda)$ has only a finite number of zeros in the finite interval Δ_0, each of finite multiplicity, a finite number of repetitions of the above procedure yields a function $\psi \in \mathfrak{D}$ for which the functional $\Phi(\psi; \lambda)$ vanishes for no $\lambda \in \Delta_0$.

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[11] Cf. M. G. Krein [5], Vol. I.

THEOREM 1: *Let Δ_0 be a finite interval and let ψ be a function in $\mathfrak{D}$ such that*

$$\Phi(\psi; \lambda) \neq 0$$

if $\lambda \in \Delta_0$. Furthermore, let E_λ be the spectral function of $\tilde{L}$ and let

$$g(t) = \int_{\Delta_0} \frac{1}{\Phi(\psi; \lambda)} \, dE_\lambda \psi.$$

Then for each function $f \in \mathfrak{D}$ and each interval $\Delta \subset \Delta_0$,

$$E(\Delta) f = \int_{\Delta} \Phi(f; \lambda) \, dE_\lambda g. \tag{4}$$

Proof: If we define the functional $F(f; \lambda)$ in terms of the direction functional $\Phi(f; \lambda)$ by the relation

$$F(f; \lambda) = \frac{\Phi(f; \lambda)}{\Phi(\psi; \lambda)},$$

then the formula to be proved is equivalent to the equation

$$E(\Delta) f = \int_{\Delta} F(f; \lambda) \, dE_\lambda \psi. \tag{4'}$$

We proceed to the proof of this equation. Assume that the left end points of the intervals Δ_0 and Δ coincide and let the point α be the common left end point of these intervals.

Consider the element $w_\mu = w_\mu(t)$ of the space $L^2(0, \infty)$ which is defined by the equation

$$w_\mu = \int_\alpha^\mu dE_\lambda f - \int_\alpha^\mu F(f; \lambda) \, dE_\lambda \psi$$

as a function of the parameter μ in the interval Δ_0. To establish equation (4′) it is necessary to prove that $w_\mu = 0$ for $\mu \in \Delta_0$ and, for this, it is obviously sufficient to show that

$$\frac{d}{d\mu} w_\mu = 0.$$

This equation must be understood in the sense of strong convergence in $L^2(0, \infty)$, i.e.,

$$\lim_{\delta \to 0} \frac{1}{\delta^2} \| w_{\mu+\delta} - w_\mu \|^2 = \lim_{\delta \to 0} \frac{1}{\delta^2} \int_0^\infty | w_{\mu+\delta}(t) - w_\mu(t) |^2 dt = 0.$$

We introduce the projection operator

$$E(\Delta_{\mu,\delta}) = E_{\mu+\delta} - E_{\mu},$$

and estimate $\| w_{\mu+\delta} - w_{\mu} \|$. We have

$$\begin{aligned} \| w_{\mu+\delta} - w_{\mu} \| &= \| E(\Delta_{\mu,\delta}) f - \int_{\mu}^{\mu+\delta} F(f;\lambda)\, dE_{\lambda}\psi \| \leqq \\ &\leqq \| E(\Delta_{\mu,\delta}) [f - F(f;\mu)\psi] \| + \\ &+ \| E(\Delta_{\mu,\delta}) F(f;\mu)\psi - \int_{\mu}^{\mu+\delta} F(f;\lambda)\, dE_{\lambda}\psi \|. \end{aligned} \tag{5}$$

We estimate the first and second summands on the right side of this inequality. Since

$$\Phi(f - F(f;\mu)\psi;\mu) = \Phi(f;\mu) - F(f;\mu)\Phi(\psi;\mu) = 0,$$

it follows by 2° that the difference $f - F(f;\mu)\psi$ can be represented in the form

$$f - F(f;\mu)\psi = \tilde{L}\varphi - \mu\varphi, \tag{6}$$

and, hence,

$$\begin{aligned} \| E(\Delta_{\mu,\delta}) [f - F(f;\mu)\psi] \|^2 &= \| E(\Delta_{\mu,\delta}) [\tilde{L}\varphi - \mu\varphi] \|^2 = \\ &= \int_{\mu}^{\mu+\delta} |\lambda - \mu|^2 d(E_{\lambda}\varphi, \varphi) \leqq \delta^2 \| E(\Delta_{\mu,\delta}\varphi \|^2. \end{aligned}$$

As $\delta \to 0$, the element $E(\Delta_{\mu,\delta})\varphi$ tends to the element $E(\Delta_{\mu,0})\varphi$ which either is zero or is an eigenfunction of the operator $\tilde{L}$ corresponding to the eigenvalue μ. In the latter case, formula (6) yields the equation

$$E(\Delta_{\mu,0}) f - F(f;\mu) E(\Delta_{\mu,0})\psi = 0.$$

This shows that formula (4′) is valid for $\Delta = \Delta_{\mu,0}$. Therefore, it was possible to assume from the beginning that $E(\Delta_{\mu,0})\varphi = 0$ which means that

$$\lim_{\delta \to 0} \frac{1}{\delta} \| E(\Delta_{\mu,\delta}) [f - F(f;\mu)\psi] \| = 0. \tag{7}$$

For the second summand on the right side of the inequality (5), we obtain

$$
\begin{aligned}
&\| E(\Delta_{\mu,\delta}) F(f;\mu)\psi - \int_{\mu}^{\mu+\delta} F(f;\lambda)\, dE_{\lambda}\psi \|^2 = \\
&= \left\| \int_{\mu}^{\mu+\delta} [F(f;\mu) - F(f;\lambda)]\, dE_{\lambda}\psi \right\|^2 \leqq \\
&\leqq \int_{\mu}^{\mu+\delta} |F(f;\mu) - F(f;\lambda)|^2 d(E_{\lambda}\psi, \psi) = \\
&= \int_{\mu}^{\mu+\delta} \left| \frac{\partial F(f;\lambda)}{\partial\lambda} \right|^2_{\lambda=\xi} (\lambda-\mu)^2 d(E_{\lambda}\psi,\psi) = M^2\delta^2(E(\Delta_{\mu,\delta})\psi,\psi),
\end{aligned} \tag{8}
$$

where

$$\mu < \xi < \mu + \delta.$$

In (8), M denotes the maximum of the absolute value of the derivative $\frac{\partial F(f;\lambda)}{\partial\lambda}$ in the interval Δ_0.

We remark that if μ is an eigenvalue of the operator $\tilde{L}$, then ψ is not an eigenfunction of this operator corresponding to μ. Otherwise,

$$\Phi(\psi;\lambda) = \frac{1}{\mu}\Phi(\tilde{L}\psi;\lambda) = \frac{\lambda}{\mu}\Phi(\psi;\lambda),$$

which implies that $\Phi(\psi;\lambda) = 0$ for $\lambda \neq \mu$, and this contradicts the choice of the function ψ.

Thus, from inequality (8) it follows that

$$\lim_{\delta\to 0} \frac{1}{\delta} \left\| E(\Delta_{\mu,\delta}) F(f;\mu)\psi - \int_{\mu}^{\mu+\delta} F(f;\lambda)\, dE_{\lambda}\psi \right\| = 0. \tag{9}$$

From (5), (7) and (9) we conclude that $w_{\mu} = 0$. This yields equation (4′), which was to be proved.

Now we come to the fundamental theorem of the present section.

THEOREM 2: *If L, $\tilde{L}$ and $u(t;\lambda)$ have the meanings given in the beginning of the present section, then the inversion formulas which are related to the operator $\tilde{L}$ have the forms*

$$\Phi(\lambda) = \int_{0}^{\infty} f(t)\, u(t;\lambda)\, dt, \tag{10}$$

$$f(t) = \int_{-\infty}^{\infty} \Phi(\lambda)\, u(t;\lambda)\, d\sigma(\lambda). \tag{11}$$

Here $\sigma(\lambda)$ *is some nondecreasing function determined by the operator* $\tilde{L}$; $\Phi(\lambda)$ *and* $f(t)$ *run through the spaces* $L^2_\sigma(-\infty, \infty)$ *and* $L^2(0, \infty)$ *respectively. If one of the functions* $\Phi(\lambda)$ *and* $f(t)$ *vanishes outside of a finite interval, then the integral on the right side of the corresponding formula can be understood in the ordinary sense. In the case of arbitrary functions* $\Phi(\lambda)$ *and* $f(t)$ *the integrals must be understood as limits in the norms of* L^2_σ *and* $L^2(0, \infty)$, *respectively, of the proper integrals*

$$\int_0^C f(t)\, u(t; \lambda)\, dt,$$

$$\int_A^B \Phi(\lambda)\, u(t; \lambda)\, d\sigma(\lambda).$$

Proof: We note first that in formula (4) it is possible to take the element $g_0 = E(\Delta_0)g$ instead of the element g. We subdivide the λ-axis into finite intervals $\Delta_k(\pm k = 0, 1, 2, \ldots)$ and we choose for each interval Δ_k an element g_k in the same way as we choose the element g_0 for the interval Δ_0. We let

$$g = \sum_{k=-\infty}^{\infty} g_k,$$

and in case the orthogonal series on the right side diverges, we understand g to be an improper element[12] (cf. Section 77) such that, for each finite interval Δ,

$$E(\Delta)\, g = \sum_k E(\Delta)\, g_k,$$

where the series on the right side contains only a finite number of summands.

If f is an arbitrary function in $\mathfrak{D}$ and Δ is an arbitrary finite interval of the λ-axis then

$$\Delta = \Delta_m^* + \sum_{k=m+1}^{n-1} \Delta_k + \Delta_n^* \qquad (\Delta_m^* \subset \Delta_m;\ \Delta_n^* \subset \Delta_n),$$

and

$$E(\Delta)f = E(\Delta_m^*)f + \sum_{k=m+1}^{n-1} E(\Delta_k)f + E(\Delta_n^* f).$$

Therefore, for each finite interval Δ of the λ-axis,

$$E(\Delta)f = \int_\Delta \Phi(f; \lambda)\, dE_\lambda g.$$

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[12] A. J. Povzner and W. A. Martschenko communicated to us a proof of the fact that for $p(t) = 1$, this element is always improper.

Let $\Delta \rightarrow [-\infty, \infty]$ to obtain the equation

$$f(t) = \int_{-\infty}^{\infty} \Phi(f; \lambda)\, dE_\lambda g. \tag{12}$$

Since the manifold $\mathfrak{D}$ is dense in $L^2(0, \infty)$, this formula implies that the operator $\tilde{L}$ has a simple spectrum and the function $g = g(t)$ is its generating element.

From the theorem of Section 69 on the canonical representation of a self-adjoint operator with simple spectrum it follows that there exists an isometric mapping

$$f(t) = \int_{-\infty}^{\infty} \Phi(\lambda)\, dE_\lambda g$$

of the space L_σ^2 where $\sigma(\lambda) = (E_\lambda g, g)$ onto the space $L^2(0, \infty)$. Under this mapping, an arbitrary function $f(t)$ in $\mathfrak{D}$ corresponds to the direction functional $\Phi(f; \lambda) = \Phi(\lambda)$. Thus, the operator $\tilde{L}$, is isomorphic to the operator of multiplication by λ in the space L_σ^2, so that

$$\tilde{L} f = \int_{-\infty}^{\infty} \lambda\, \Phi(\lambda)\, dE_\lambda g$$

for $f(t) \in D_{\tilde{L}}$.

We turn now to the derivation of formulas (10) and (11). If $f(t)$ is an arbitrary function in $L^2(0, \infty)$ and

$$f_n(t) = \begin{cases} f(t) & (t < n), \\ 0 & (t \geqq n), \end{cases}$$

then, by (12)

$$f_n(t) = \int_{-\infty}^{\infty} \Phi(f_n; \lambda)\, dE_\lambda g. \tag{13}$$

Let

$$\Phi_n(\lambda) = \Phi(f_n; \lambda) = \int_0^n f(t)\, u(t; \lambda)\, dt. \tag{14}$$

Since the sequence of functions $\{f_n(t)\}$ converges in the norm of $L^2(0, \infty)$ to the function $f(t)$, the isometric correspondence between the spaces $L^2(0, \infty)$ and L_σ^2 implies that the sequence $\{\Phi(\lambda)\}$ converges in the norm of L_σ^2 to some function $\Phi(\lambda)$. Passing to the limit in the sense of the indicated norm in equations (13) and (14), we obtain the formulas

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \Phi(\lambda)\, dE_\lambda g, \\ \Phi(\lambda) &= \int_{0}^{\infty} f(t)\, u(t; \lambda)\, dt. \end{aligned} \tag{15}$$

The second of these formulas coincides with (10) and, therefore, it remains only to prove formula (11).

Obviously, it is sufficient to prove that formula (11) is the inversion of formula (10) if $\Phi(\lambda)$ runs through the manifold $\mathfrak{M}$ of functions in L_σ^2 which vanish outside of a finite interval. Thus, let the function $\Phi(\lambda)$ belong to $\mathfrak{M}$ and let $f(t)$ be the function of $L^2(0, \infty)$ corresponding to it, which is defined by formula (15). We introduce the function

$$\eta_t(s) = \begin{cases} 1 & (s \leqq t), \\ 0 & (s > t), \end{cases}$$

and use the fact that the correspondence between $L^2(0, \infty)$ and L_σ^2 is isometric to obtain the equation

$$\int_0^\infty f(s)\, \eta_t(s)\, ds = \int_{-\infty}^{\infty} \Phi(\lambda)\, \Phi(\eta_t; \lambda)\, d\sigma(\lambda),$$

or

$$\int_0^t f(s)\, ds = \int_{-\infty}^{\infty} \Phi(\lambda) \left\{ \int_0^t u(s; \lambda)\, ds \right\} d\sigma(t).$$

The left member has the derivative $f(t)$ almost everywhere and the right integral in λ may be expressed with finite limits since $\Phi(\lambda) \in \mathfrak{M}$. Therefore, the derivative with respect to t of the right member is

$$\int_{-\infty}^{\infty} \Phi(\lambda)\, u(t; \lambda)\, d\sigma(\lambda),$$

and the proof is complete.

Our result concerning the inversion formula would be incomplete if we did not show how to actually find the nondecreasing function $\sigma(\lambda)$. We now concern ourselves with this question; with this aim we denote by $u(t; \lambda)$ that solution of the equation

$$l[y] - \lambda y = 0,$$

which for $t = 0$ satisfies the conditions which characterize the given extension $\tilde{L}$ of the operator L. As was indicated above, this condition has the form

$$p(t)\,\varphi'(t)\,|_{t=0} = \theta\varphi(0).$$

The real parameter θ we take as finite so that $u(0, \lambda) \neq 0$ and we can let

$$u(0; \lambda) = 1.$$

Furthermore, we denote by $v(t; \lambda)$ a second solution of the homogeneous equation which is normalized by means of the equations

$$v(0; \lambda) = 0, \quad p(t)\,v'(t; \lambda)\,|_{t=0} = -1.$$

The kernel $K(t, s; \lambda)$ of the integral operator which is the resolvent of the operator $\tilde{L}$, i.e., the Green's function, has the form

$$(16) \quad K(t,s;z) = \begin{cases} u(s;z)\,[v(t;z) + m(z)u(t;z)\,] & (s \leqq t), \\ u(t;z)\,[v(s;z) + m(z)\,u(s;z)\,] & (s > t), \end{cases} \quad (\Im z > 0).$$

The function $m(z)$ is uniquely defined by the condition

$$v(t; z) + m(z)\,u(t; z) \in L^2(0, \infty).$$

We now let $\varphi(t)$ run through the set of all functions in $D_{\tilde{L}}$ which vanish outside of a finite interval. Let

$$(\tilde{L} - zE)\,\varphi = f,$$

where z $(\Im z > 0)$ is fixed. The set $\mathfrak{N}$ through which the function $f(t)$ runs obviously is dense in $L^2(0, \infty)$. If $g(t)$ is an arbitrary function in $\mathfrak{D}$, then

$$\int_0^\infty \varphi(t)\overline{g(t)}\,dt = \int_{-\infty}^\infty \Phi(\varphi; \lambda)\,\overline{\Phi(g; \lambda)}\,d\sigma(\lambda).$$

But since

$$\varphi(t) = \int_0^\infty K(t, s; z) f(s)\,ds,$$

and, by property 3° of the direction functional,

$$\Phi(\varphi; \lambda) = \frac{\Phi(f; \lambda)}{\lambda - z},$$

we have

$$\int_0^\infty \int_0^\infty K(t, s; z) f(s)\overline{g(t)}\,ds\,dt = \int_{-\infty}^\infty \frac{\Phi(f; \lambda)\,\overline{\Phi(g; \lambda)}}{\lambda - z}\,d\sigma(\lambda).$$

This equation is valid for all $f, g \in \mathfrak{D}$, since $\mathfrak{N}$ is dense in $L^2(0, \infty)$. Let

$$f(t) = g(t) = \begin{cases} \frac{1}{\delta} \text{ for } 0 \leqq t \leqq \delta, \\ 0 \text{ for } t > \delta. \end{cases}$$

Then our equation takes the form

$$\frac{1}{\delta^2}\int_0^\delta\int_0^\delta K(t,s;z)\,dt\,ds = \int_{-\infty}^{\infty} \omega_\delta(\lambda)\frac{d\sigma(\lambda)}{\lambda - z}, \tag{17}$$

where

$$\omega_\delta(\lambda) = \left[\frac{1}{\delta}\int_0^\delta u(t;\lambda)\,dt\right]^2.$$

As $\delta \to 0$, the left member of (17) tends to the limit

$$K(0, 0; z) = m(z).$$

Therefore,

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} \omega_\delta(\lambda)\frac{d\sigma(\lambda)}{\lambda - z} = m(z),$$

which yields

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} \omega_\delta(\lambda)\frac{d\sigma(\lambda)}{\lambda^2 - 1} = \mathfrak{Im}\, m(i).$$

It is easy to conclude that the integral

$$\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda^2 + 1}$$

exists and equals $\mathfrak{Im}\, m(i)$. Therefore,

$$\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - z} - \frac{1}{\lambda - i}\right)d\sigma(\lambda) = m(z) - m(i).$$

It follows that

$$m(z) = \mathfrak{Re}\, m(i) + \int_{-\infty}^{\infty} \frac{1 + \lambda z}{\lambda + z}\,\frac{d\sigma(\lambda)}{\lambda^2 + 1}. \tag{18}$$

Applying the Stieltjes inversion formula (cf. Section 59), we obtain

$$\frac{\sigma(\lambda - 0) + \sigma(\lambda + 0)}{2} = \text{const.} + \lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \mathfrak{Im}\, m(x + iy)\,dx.$$

It is possible to show that this formula is valid in the case when the boundary condition has the form $\varphi(0) = 0$.

Completing the present section, we make several remarks concerning the case when the deficiency indices of the operator are (2, 2). A class of self-adjoint extensions of such an operator which was first studied by Weyl in the paper quoted on page 186 will be discussed.

Let L have the deficiency indices (1, 1) or (2, 2). Let L_θ denote an extension of L determined by the boundary condition

$$(19) \qquad p(t)\,\varphi'(t)\,|_{t=0} = \theta\varphi(0),$$

so that D_{L_θ} consists of all the functions $\varphi(t) \in D_{L^*}$ which satisfy condition (19). Obviously L_θ is a symmetric extension of L. If the deficiency indices of the operator L are (1, 1), then the operator L_θ is self-adjoint. Letting θ vary from $-\infty$ to ∞, we obtain all self-adjoint extensions of L. If the deficiency indices of the operator L are (2, 2), then the deficiency indices of the operator L_θ will be (1, 1). Each self-adjoint extension $\tilde{L}_\theta$ of the operator L_θ is an extension of L. Varying θ in the interval $(-\infty, \infty)$ and taking for each value θ all possible extensions $\tilde{L}_\theta$, we obtain a class of self-adjoint extensions of L. This class is characterized by the fact that one of the two boundary conditions (2) of Section 5 which define the self-adjoint extension has the form (19); obviously, the second of these conditions involves only the singular end point $t = \infty$. Thus, the class of self-adjoint extensions obtained are characterized by *reduced conditions*.

The assumption made in the beginning of this section that the deficiency indices of the operator L are (1, 1) was used only once, namely, in the proof of property 2° of the direction functional. However, it is easy to see that this property 2° is maintained in the case of the deficiency indices (2, 2), if only those self-adjoint extensions $\tilde{L}$ are considered which are characterized by the reduced conditions. We denote such extensions by $\tilde{L}_\theta$.

Thus, all of the above propositions remain valid for extensions $\tilde{L}_\theta$ of the operator L with the deficiency indices (2, 2). In particular, from Theorem 2 it follows that the self-adjoint extensions $\tilde{L}_\theta$ have simple spectra.

It is not difficult to see that as in the case of the deficiency indices (1, 1), the kernel of the resolvent $\tilde{R}_z$ of the operator $\tilde{L}_\theta$ (for $\theta \neq \infty$) is defined by the formula (16) where the function $m(z)$ is uniquely determined by the choice of the self-adjoint extension $\tilde{L}_\theta$ of L_θ. In particular, $K(0, 0; z) = m(z)$. Reasoning further as in the case of the deficiency indices (1, 1) we obtain formula (18). Since, by Theorem 2 of Section 6, the operator $\tilde{L}_\theta$ has a pure point spectrum with the unique limit point at infinity, the function $\sigma(\lambda)$ is piecewise constant and formula (18) shows that $m(z)$ is a meromorphic function of z.

In correspondence with the results of Section 4 of Appendix I, the scalar product $(\tilde{R}_z f, f)$ for fixed $f(t) \in L^2(0, \infty)$ and z ($\Im z > 0$) runs through a circle $C(f; z)$ as $\tilde{R}_z$ runs through the set of all orthogonal resolvents of the operator L_θ. But from formula (16) it follows that the scalar product $(\tilde{R}_z f, f)$ has the form $\alpha + \beta m(z)$, where the constants α and β do not depend on the choice of the resolvents. Therefore, as the point $(\tilde{R}_z f, f)$ runs through the circle $C(f; z)$, the point $\omega = m(z)$ also runs through a certain circle which is called the *Weyl limit circle*. In the case of the deficiency indices (1, 1) (and only in that case) the Weyl limit circle degenerates to a point. Now we clarify the meaning of the terms "*limit circle*" and "*limit point*" mentioned in the end of Section 5 of Appendix II. We do not present here the method by which Weyl arrived at the limit circle.[13]

We remark that although the circle $C(f; z)$ depends on the choice of f and z, the corresponding limit circle C depends only on z but does not depend on f, because the function $m(z)$ does not depend on f. We shall find the equation of the limit circle. With this aim, following Krein, we use the Hilbert functional equation for the resolvent of the operator L_θ which yields, without difficulty,

$$K(t,s;z) - K(t,s;\bar{z}) = (z-\bar{z}) \int_0^\infty K(t,\xi;z)\,K(\xi,s;\bar{z})\,d\xi.$$

Let $t = s = 0$ and use formula (16) to obtain

$$m(z) - \overline{m(z)} = (z-\bar{z}) \int_0^\infty |\,v(\xi;z) + m(z)\,u(\xi;z)\,|^2 d\xi.$$

Hence, the equation of the limit circle has the form

$$\int_0^\infty |\,v(\xi;z) + \omega u(\xi;z)\,|^2 d\xi = \frac{\omega - \bar{\omega}}{z - \bar{z}}.$$

The case $\theta = \infty$ involves a modification of the above arguments and leads to a similar result. We shall not go into this further.

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[13] Cf. B. M. Levitan [2], Vol. I.

8. Generalization to Differential Operators of Arbitrary Order

In Section 72 we saw that every self-adjoint operator A with a spectrum of multiplicity $r < \infty$ determines an isometric mapping of H onto the space of vector functions L_S^2. Under this mapping the operator A goes into the operator of multiplication by the independent variable. We recall that the matrix distribution function $S(\lambda)$ is determined by the spectral function E_λ of the operator A if one chooses some generating basis $g_1, g_2, \dots, g_p$ $(r \leqq p < \infty)$ of this operator and lets

$$S(\lambda) = \left((E_\lambda g_i, g_k) \right)_{j,k=1}^{p}.$$

The indicated isometric mapping is defined by the formulas

$$\overrightarrow{\Phi(\lambda)} = Vf, \tag{1}$$

$$f = V^{-1}\overrightarrow{\Phi(\lambda)} = \mathrm{l.i.m.} \int_{-N}^{N} \sum_{k=1}^{p} \Phi_k(\lambda)\, dE_\lambda g_k, \tag{2}$$

where the element f and the vector function $\overrightarrow{\Phi(\lambda)} = \{\Phi_1(\lambda), \dots, \Phi_p(\lambda)\}$ run through the spaces H and L_S^2 respectively.

Formulas (1) and (2) we call the *inversion formulas* related to the self-adjoint operator A. We remark that this definition is more general than the definition given in the preceding section for the case of operators with a simple spectrum since it does not require that the basis which generates the distribution function $S(\lambda)$ be minimal. In particular, by the definition of the present section, an operator with a simple spectrum has related inversion formulas which define an isometric correspondence between H and L_S^2, where the order of the matrix $S(\lambda)$ is larger than one.

We now derive the inversion formulas for quasi-differential operators of arbitrary order. Let L be a quasi-differential operator with minimal domain, generated by the operation l of order $2n$ on the interval $(0, \infty)$ with a singular end point and let $\tilde{L}$ be an arbitrary self-adjoint extension of the operator L. We make no assumptions now about the deficiency indices of the operator L.

We introduce on the manifold $\mathfrak{D}$ of functions of the space $L^2(0, \infty)$ which vanish outside a finite interval, the $2n$ direction functionals

$$\Phi_j(f; \lambda) = \int_0^\infty f(t)\, u_j(t; \lambda)\, dt \qquad (j = 1, 2, \dots, 2n), \tag{3}$$

where $\{u_j(t;\lambda)\}_{j=1}^{2n}$ is the fundamental system of solutions of the equation

$$l[u]-\lambda u=0,$$

which satisfy the initial conditions

$$u_j^{[k-1]}(0;\lambda)=\begin{cases}1 & (j=k)\\ 0 & (j\neq k)\end{cases}\quad (j,k=1,2,\ldots,2n).$$

It is easy to verify that each of the functionals $\Phi_j(f;\lambda)$ has properties 1° and 3° of Section 7. As far as property 2° is concerned, it is now formulated in the following form.

2° If for some function $f\in\mathfrak{D}$ and some real value λ the $2n$ equations

$$\Phi_j(f;\lambda)=0\qquad (j=1,2,\ldots,2n)$$

hold, then the equation

$$(\tilde{L}-\lambda E)\varphi=f$$

has a solution in the class of functions which vanish outside a finite interval.

LEMMA 1:[14] *For each finite interval Δ_0 of the λ-axis there exists a system of functions $\psi_1(t),\psi_2(t),\ldots,\psi_{2n}(t)$ in $\mathfrak{D}$ such that the determinant*

$$\mathrm{D}(\lambda)=\mathrm{Det}\,(\Phi_j(\psi_k;\lambda))_{j,k=1}^{2n}$$

does not vanish in the interval Δ_0.

The proof of the lemma is easy to outline by the pattern of the proof of the corresponding lemma of Section 7 which is based on the analyticity of the determinant $\mathrm{D}(\lambda)$.

THEOREM 1: *Let Δ_0 be a finite interval and let $\psi_1,\psi_2,\ldots,\psi_{2n}$ be a system of functions in $\mathfrak{D}$ such that*

$$\mathrm{Det}\,(\Phi_j(\psi_k;\lambda))\neq 0\qquad (\lambda\in\Delta_0).$$

Furthermore, let E_λ be the spectral function of the operator $\tilde{L}$ and let $g_1,g_2,\ldots,g_{2n}$ be a system of functions defined by the equations

$$g_i(t)=\int_{\Delta_0}\sum_{k=1}^{2n}\Omega_{ik}(\lambda)\,dE_\lambda\psi_k,$$

where $\Omega_{ik}(\lambda)$ are the elements of the matrix which is the inverse of the matrix

$$(\Phi_j(\psi_k;\lambda))_{j,k=1}^{2n}.$$

Then for each function $f\in\mathfrak{D}$ and each interval $\Delta\subset\Delta_0$,

$$E(\Delta)f=\int_{\Delta}\sum_{k=1}^{2n}\Phi_j(f;\lambda)\,dE_\lambda g_j. \tag{4}$$

---

[14] This and the following proposition are due to M. G. Krein (cf. the work cited in the beginning of Section 7).

The proof of this theorem is completely analogous to the proof of Theorem 1 of Section 7. In the same way as there, we replace the condition (4) to be proved by the equation

$$E(\Delta)f = \int_{\Delta} \sum_{j=1}^{2n} F_j(f;\lambda)\, dE_\lambda \psi_j,$$

where

$$F_j(f;\lambda) = \sum_{k=1}^{2n} \Omega_{jk}(\lambda)\, \Phi_k(f;\lambda),$$

and we introduce the element

$$w_\mu = \int_a^\mu dE_\lambda f - \int_a^\mu \sum_{j=1}^{2n} F_j(f;\lambda)\, dE_\lambda\, \psi_j.$$

Inequality (5) of Section 7 now has the form

$$\|w_{\mu+\delta} - w_\mu\| \leqq \left\| E(\Delta_{\mu,\delta}) \left[f - \sum_{j=1}^{2n} F_j(f;\mu)\,\psi_j \right] \right\| +$$
$$+ \left\| E(\Delta_{\mu,\delta}) \sum_{j=1}^{2n} F_j(f;\mu)\,\psi_j - \int_\mu^{\mu+\delta} \sum_{j=1}^{2n} F_j(f,\lambda)\, dE_\lambda\, \psi_j \right\|.$$

The first summand on the right side of the inequality is estimated in the same way as in Section 7. As far as the second summand is concerned,

$$(5) \qquad \sum_{j=1}^{2n} \left\| E(\Delta_{\mu,\delta})\, F_j(f;\mu)\,\psi_j - \int_\mu^{\mu+\delta} F_j(f;\lambda)\, dE_\lambda \psi_j \right\|^2 \leqq$$
$$\leqq \sum_{j=1}^{2n} M_j^2 \delta^2 (E(\Delta_{\mu,\delta})\,\psi_j, \psi_j),$$

where M_j denotes the maximum of the absolute value of the derivative $\dfrac{\partial F_j}{\partial \lambda}$ in the interval Δ_0. If μ is an eigenvalue of the operator $\tilde{L}$, then the function ψ_k is not an eigenfunction belonging to this operator since, otherwise, by property (3),

$$\Phi_j(\psi_k;\lambda) = \frac{1}{\mu}\, \Phi_j(\tilde{L}\psi_k;\lambda) = \frac{\lambda}{\mu}\, \Phi_j(\psi_k;\lambda) \quad (j = 1, 2, \ldots, 2n),$$

which would imply that $\Phi_j(\psi_k;\lambda) = 0$ for $\lambda \neq \mu$ and, hence, D $(\lambda) = 0$ for $\lambda \neq \mu$. This contradicts the choice of the system $\psi_1, \psi_2, \ldots, \psi_{2n}$.

Thus, the right side of inequality (5) has magnitude greater than δ^2, which yields the validity of the theorem.

THEOREM 2: *If L, $\tilde{L}$ and $u_j(t;\lambda)$ $(j=1, 2, \ldots, 2n)$ have the meanings given in the beginning of the present section, then there exists a matrix distribution function $S(\lambda) = (\sigma_{ik}(\lambda))_{i,k=1}^{n}$ for which the formulas*

$$\Phi_j(\lambda) = \mathrm{l.i.m.} \int_{-N}^{N} f(t)\, u_j(t;\lambda)\, dt \qquad (j = 1, 2, \ldots, 2n),$$

$$f(t) = \mathrm{l.i.m.} \int_{-N}^{N} \sum_{i,k=1}^{2n} \Phi_i(\lambda) u_k(t;\lambda)\, d\sigma_{ik}(\lambda)$$

establish an isometric mapping of the space $L^2(0, \infty)$ onto the space of vector functions L_S^2.

Proof: We substitute $\Delta = \Delta_0$ in formula (4) and we replace the system of elements $\{g_j\}_{j=1}^{2n}$ by the system $g_j^{(0)} = E(\Delta_0)\, g_j$ $(j=1, 2, \ldots, 2n)$. Then we subdivide the λ-axis into finite intervals $\Delta_k(\pm k=0, 1, 2, \ldots)$ and we choose for each of these a system of elements $\{g_j^{(k)}\}_{j=1}^{2n}$ in the same way as the system $\{g_j^{(0)}\}_{j=1}^{2n}$ was chosen for the interval Δ_0.

Let

$$g_i = \sum_{k=-\infty}^{\infty} g_i^{(k)} \qquad (i = 1, 2, \ldots, 2n).$$

Then, as in Section 7, we obtain a system of elements (proper or improper) $g_1, g_2, \ldots, g_{2n}$ which satisfy the condition

$$E(\Delta)\, g_i = \sum_{k} E(\Delta)\, g_i^{(k)}$$

for each finite interval Δ. (The series on the right contains only a finite number of summands.) Furthermore, for each finite interval Δ of the λ-axis we obtain the formula

$$E(\Delta) f = \int_{\Delta} \sum_{j=1}^{2n} \Phi_j(f;\lambda)\, dE_\lambda g_j,$$

and for $\Delta = [-\infty, \infty]$ we obtain

$$f = \mathrm{l.i.m.} \int_{-N}^{N} \sum_{j=1}^{2n} \Phi_j(f;\lambda)\, dE_\lambda g_j.$$

Since the manifold $\mathfrak{D}$ is dense in $L^2(0, \infty)$, the last formula implies that the multiplicity of the spectrum of the operator $\tilde{L}$ does not exceed $2n$ and that $g_1, g_2, \ldots, g_{2n}$ is a generating basis.

The closure relation for $f \in \mathfrak{D}$ has the form

$$\int_{0}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \sum_{i,k=1}^{2n} \Phi_j(f; \lambda)\, \Phi_k(f; \lambda)\, d\sigma_{ik}(\lambda).$$

All further calculations mentioned in the end of the preceding section are carried over without difficulty to the case now under consideration. Hence, Theorem 2 is established.

9. Examples

I. The Trigonometric Functions. The differential operator:

$$l = -\frac{d^2}{dt^2} \qquad (0 \leqq t < \infty).$$

The deficiency indices of the operator L are $(1, 1)$. The boundary condition which characterizes a self-adjoint extension of the operator are

$$y'(0) = \theta y(0).$$

The fundamental system of solutions:

$$u(t; \lambda) = \cos(\sqrt{\lambda} t) + \frac{\theta}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t),$$

$$v(t; \lambda) = -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t).$$

The determination of the function $m(z)$ $(\Im z > 0)$:

$$v(t;z) + m(z) u(t;z) =$$

$$= m(z)\left\{\cos(\sqrt{\lambda}\, t) + \frac{\theta}{\sqrt{z}} \sin(\sqrt{\lambda}\, t)\right\} - \frac{1}{\sqrt{z}} \sin(\sqrt{z}\, t) =$$

$$= \left\{m(z)\left[\frac{1}{2} - \frac{\theta}{2i\sqrt{z}}\right] + \frac{1}{2i\sqrt{z}}\right\} e^{-i\sqrt{z}t} +$$

$$+ \left\{m(z)\left[\frac{1}{2} + \frac{\theta}{2i\sqrt{z}}\right] - \frac{1}{2i\sqrt{z}}\right\} e^{i\sqrt{z}t};$$

the condition that this belongs to $L^2(0, \infty)$ yields

$$m(z)\left[\frac{1}{2} - \frac{\theta}{2i\sqrt{z}}\right] + \frac{1}{2i\sqrt{z}} = 0,$$

so that

$$m(z) = \frac{1}{\theta - i\sqrt{z}}.$$

The determination of $\sigma(\lambda)$:

$$\sigma(\lambda) = \text{const.} + \lim_{y\to 0} \frac{1}{\pi} \int_0^{\lambda} \mathfrak{I} \frac{1}{\theta - i\sqrt{x+iy}}\, dx\,;$$

if $\theta \geqq 0$, then

$$\sigma'(\lambda) = \frac{\sqrt{\lambda}}{\pi(\lambda+\theta^2)} \qquad (\lambda \geqq 0),$$

$$\sigma(\lambda) = 0 \qquad (\lambda < 0);$$

if $\theta < 0$, then

$$\sigma'(\lambda) = \frac{\sqrt{\lambda}}{\pi(\lambda+\theta^2)} \qquad (\lambda \geqq 0),$$

and at the point

$$\lambda_0 = -\theta^2$$

there is a jump

$$\sigma(\lambda_0 + 0) - \sigma(\lambda_0) = -2\theta.$$

The inversion formulas:

(a) $(\theta \geqq 0)$

$$\Phi(\lambda) = \int_0^{\infty} f(t) \left\{ \cos(\sqrt{\lambda}\, t) + \frac{\theta}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t) \right\} dt \quad (\lambda \geqq 0),$$

$$f(t) = \frac{1}{\pi} \int_0^{\infty} \Phi(t) \left\{ \cos(\sqrt{\lambda}\, t) + \frac{\theta}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t) \right\} \frac{\sqrt{\lambda}}{\lambda+\theta^2}\, d\lambda;$$

(b) $(\theta < 0)$

$$\Phi(\lambda) = \begin{cases} \displaystyle\int_0^{\infty} f(t) \left\{ \cos(\sqrt{\lambda}\, t) + \frac{\theta}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t) \right\} dt & (\lambda \geqq 0), \\ \displaystyle\int_0^{\infty} f(t) e^{\theta t}\, dt & (\lambda = -\theta^2), \\ 0 & (\lambda \neq -\theta^2, \lambda < 0), \end{cases}$$

$$f(t) = -2\theta\Phi(-\theta^2) e^{\theta t} + \frac{1}{\pi} \int_0^{\infty} \Phi(\lambda) \left\{ \cos(\sqrt{\lambda}\, t) + \frac{\theta}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, t) \right\} \frac{\sqrt{\lambda}}{\lambda+\theta^2}\, d\lambda.$$

If the boundary condition is of the form

$$y(0) = 0,$$

then the inversion formulas are obtained by letting $\theta \to \infty$:

$$\Phi(\lambda) = \int_0^\infty f(t) \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}}\, dt,$$

$$f(t) = \frac{1}{\pi} \int_0^\infty \Phi(\lambda) \sin(\sqrt{\lambda}\, t)\, d\lambda.$$

In this case as for $\theta = 0$, we obtain the ordinary Fourier-Plancherel transformation for the semi-axis.

II. The Legendre functions. The differential operation:

$$l = -\frac{d}{dt}(1 - t^2)\frac{d}{dt} \qquad (-1 < t < 1).$$

Both endpoints of the interval are singular.

An orthonormal system of solutions of the equation

$$(L^* - \lambda E)\, u = 0$$

for nonreal λ has the form

$$\left.\begin{aligned} u_1(t;\lambda) &= A_\mu\{P_\mu(t) + P_\mu(-t)\}, \\ u_2(t;\lambda) &= B_\mu\{P_\mu(t) - P_\mu(-t)\}, \end{aligned}\right. \qquad (A_\mu, B_\mu > 0),$$

where μ is any root of the equation

$$\mu(\mu + 1) = \lambda,$$

and $P_\mu(t)$ is a Legendre function of the first kind which can be defined by means of the following series[15]

$$P_\mu(t) = \\ = 1 + \sum_{k=1}^{\infty} \frac{(\mu+1)(\mu+2)\dots(\mu+k)(-\mu)(-\mu+1)\dots(-\mu+k-1)}{k!^2} \left(\frac{1}{2} - \frac{t}{2}\right)^k.$$

It converges for t in the circle $|\, t - 1 \,| < 2$. This series is the hypergeometric function

$$F\left(\mu + 1, -\mu, 1; \frac{1-t}{2}\right).$$

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[15] Cf. E. T. Whittaker and N. G. Watson [1], Vol. I.

Furthermore,

$$P'_\mu(t)=\frac{\mu(\mu+1)}{2}F\left(\mu+2,-\mu+1,2;\frac{1-t}{2}\right).$$

Therefore, as $t\to-1+0$,

$$\frac{P_\mu(t)}{\ln\dfrac{1-t}{1+t}}\to-\frac{\sin\pi\mu}{\pi},$$

$$(1-t^2)P'_\mu(t)\to\frac{2\sin\pi\mu}{\pi}.$$

It follows that both solutions belong to $L^2(-1,1)$, i.e., the deficiency indices of the operator L are (2, 2).

We let

$$\varphi^*(t)=\varphi(t)-\frac{1}{2}\varphi^{[1]}(t)\ln\frac{1+t}{1-t},$$

where the quasi-derivative $\varphi^{[1]}(t)$ equals

$$(1-t^2)\,\varphi'(t),$$

and then we can express the Lagrange bilinear form by

$$[\varphi,\psi]_t=\varphi^{[1]}(t)\,\overline{\psi^*(t)}-\varphi^*(t)\,\overline{\psi^{[1]}(t)}.$$

For every function $\varphi(t)\in D_{L^*}$ there exist the finite limits

$$\lim_{t\to\pm1}\varphi^{[1]}(t),\qquad\lim_{t\to\pm1}\varphi^*(t).$$

The manifold D_L is the set of all those functions $\varphi(t)\in D_{L^*}$ for which these four limits equal zero. These two results follow from the fact that for each function $f(t)\in L^2(-1,1)$, the general integral of the equation

$$l[y]=f(t)$$

has the form

$$y(t)=\left\{C_1+\frac{1}{2}\int_{-1}^{t}f(s)\ln\frac{1+s}{1-s}ds\right\}+\left\{C_2-\frac{1}{2}\int_{-1}^{t}f(s)\,ds\right\}\ln\frac{1+t}{1-t}.$$

In order to obtain a self-adjoint extension of the operator L it is necessary, by the general theory, to choose some λ in the upper half-plane (we take $\lambda=\mu(\mu+1)$, where μ is purely imaginary) and, choosing a unitary matrix $(\theta_{ik})_{i,k=1}^{2}$, to let

$$w_1(t;\lambda)=u_1(t;\lambda)+\theta_{11}\overline{u_1(t;\lambda)}+\theta_{12}\overline{u_2(t;\lambda)},$$

$$w_2(t;\lambda)=u_2(t;\lambda)+\theta_{21}\overline{u_1(t;\lambda)}+\theta_{22}\overline{u_2(t;\lambda)}.$$

Furthermore, it is necessary to find the values

$$w_k^{[1]}(\pm 1;\lambda),\ w_k^{*}(\pm 1;\lambda) \qquad (k=1,2),$$

and, in terms of them, to express the boundary conditions

$$[\varphi, w_k]_{-1}^{1}=0 \qquad (k=1,2)$$

which distinguish the domain D_{L^*} from the domain D_{L_θ}.

Using the formulas mentioned it is not difficult to establish the following table:

$$w_1^{[1]}(\pm 1)=\frac{2\sin\pi\mu}{\pi}\,\{\mp A_\mu(1-\theta_{11})-B_\mu\theta_{12}\},$$

$$w_2^{[1]}(\pm 1)=\frac{2\sin\pi\mu}{\pi}\,\{\pm A_\mu\theta_{21}+B_\mu(1-\theta_{22})\},$$

$$w_1^{*}(\pm 1) = A_\mu\{[1+\gamma(\mu)]+\theta_{11}[1+\overline{\gamma(\mu)}]\}\pm\theta_{12}B_\mu[1-\overline{\gamma(\mu)}],$$

$$w_2^{*}(\pm 1)=\theta_{21}A_\mu[1+\overline{\gamma(\mu)}]\pm B_\mu\{[1-\gamma(\mu)]+\theta_{22}[1-\overline{\gamma(\mu)}]\}.$$

We do not calculate the value of $\gamma(\mu)$, but from what follows it will be clear that $\gamma(\mu)\neq\pm 1$.

We obtain the simplest and most important self-adjoint extension by letting

$$\theta_{11}=\theta_{22}=1,\ \theta_{12}=\theta_{21}=0.$$

Then

$$w_1^{[1]}(\pm 1)=0,\ w_2^{[1]}(\pm 1)=0,$$

and

$$w_1^{*}(\pm 1) = A_\mu[2+\gamma(\mu)+\overline{\gamma(\mu)}],$$

$$w_2^{*}(\pm 1) = \pm B_\mu[2-\gamma(\mu)-\overline{\gamma(\mu)}].$$

Since neither $w_1(t)$ nor $w_2(t)$ belong to D_L, the last expressions are distinct from zero, which implies that $\gamma(\mu)\neq\pm 1$.

The boundary conditions which characterize the extension considered have the form

$$(1)\qquad \varphi^{[1]}(1)=\varphi^{[1]}(-1)=0.$$

It is possible to express them differently. First, they are equivalent to the condition

$$\frac{1}{\sqrt{1-t^2}}\,\varphi^{[1]}(t)\in L^2(-1,1).$$

Second, they are equivalent to the following requirement:

(3) $\varphi(t)$ tends to finite limits as $t \to \pm 1$.

We now show that these conditions are equivalent. It is sufficient to consider the case of real functions. Then

$$(4)\quad \begin{aligned} &-\int_\alpha^\beta \varphi(t)\,[(1-t^2)\,\varphi'(t)]'\,dt = \\ &= -(1-t^2)\,\varphi'(t)\,\varphi(t)\,\big|_\alpha^\beta + \int_\alpha^\beta (1-t^2)\,\varphi'^2(t)\,dt \end{aligned} \qquad (-1<\alpha<\beta<1),$$

the left side of which has finite limits for $\alpha \to -1$ and $\beta \to 1$, for each function $\varphi(t) \in D_{L^*}$. On the other hand,

$$\varphi^{[1]}(t) = \varphi^{[1]}(-1) - \int_{-1}^{t} l\,[\varphi(s)\,]\,ds = \varphi^{[1]}(1) + \int_{1}^{t} l\,[\varphi(s)]\,ds.$$

Therefore, condition (1) is satisfied, so that

$$|\,\varphi^{[1]}(t)\,| \leqq \sqrt{1 \pm t}\,\|\,L^*\varphi\,\|,$$

and, consequently,

$$\lim_{t\to\pm 1} \varphi^{[1]}(t)\ln\frac{1+t}{1-t} = 0.$$

It follows that, as $t \to \pm 1$, $\varphi(t)$ tends to finite limits, i.e., condition (3) is satisfied. From the identity (4) follows also the existence of the integral[16]

$$(5)\qquad I(\varphi) = \int_{-1}^{1} (1-t^2)\,\varphi'^2(t)\,dt$$

so that condition (2) is satisfied. Thus, (1) → (3) and (1) → (2). It remains to prove that (2) → (1). But if the integral (5) exists, then $\varphi^{[1]}(1)$ cannot be different from zero, since otherwise the finiteness of $\varphi^*(1)$ would imply that $\varphi(t)$ has an infinite limit as $t \to 1$. This would contradict identity (4).

The self-adjoint extension characterized by any one of the conditions (1), (2), and (3) reduces to the classical expansion in terms of Legendre polynomials. Since the boundary conditions (1) are reduced in the sense defined earlier, the spectrum of the extension is simple. The spectrum is discrete and the inverse operator is completely continuous. The Legendre polynomials

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[16] We remark that the equation $l\,[\varphi]=0$ coincides with the Euler equation for the integral $I(\varphi)$. In this connection cf. Friedrichs [2], Vol. I.

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n(t^2-1)^n}{dt^n} \qquad (n = 0, 1, 2, \ldots)$$

satisfy the equations

$$-\frac{d}{dt}(1 - t^2)\frac{du}{dt} - n(n + 1)u = 0$$

and, in addition, they satisfy condition (3). But, since the set of all polynomials $P_n(t)$ form a complete orthogonal system in $L^2(-1, 1)$, the spectrum of the extension considered consists of the points

$$n(n + 1) \qquad (n = 0, 1, 2, \ldots),$$

and the inversion formulas corresponding to this case follow from the expansion in the series of Legendre polynomials.

III. The Tchebysheff-Hermite functions are connected with the differential operation

$$l = -\frac{d^2}{dt^2} + t^2 \qquad (-\infty < t < \infty), \tag{6}$$

which, in particular, is very important in the theory of so-called linear oscillators.

For the deficiency indices of the operator L we use Theorem 4 of Section 6, applying this theorem to an operation of the more general form

$$-\frac{d^2}{dt^2} + q(t) \tag{7}$$

where $q(t) \geqq 0$.

We apply the method of successive approximations to the equation

$$u'' = q(t)u \qquad (0 \leqq t < \infty),$$

and this shows that the solution $u(t)$ which equals one for $t = 0$ satisfies the inequality

$$u(t) \geqq 1$$

for all $t \geqq 0$, and hence does not belong to $L^2(0, \infty)$.

Thus, the equation

$$-u'' + q(t)u - \lambda u = 0$$

has for $\lambda = 0$ a solution which does not belong to $L^2(0, \infty)$. By Theorem

4 of Section 6, the operator with minimal domain which is generated by the operator (7) on the semi-axis $(0, \infty)$ has deficiency indices (1,1). Obviously, the operator with minimal domain generated by the operator (7) on the negative semi-axis also has deficiency indices (1, 1). Applying Theorem 3 of Section 4, we find that the operator with minimal domain which is generated by operation (7) on the whole axis has deficiency indices (0, 0).

Obviously we would obtain the same result if we had replaced the requirement of the non-negativity of the function $q(t)$ by the condition of its semi-boundedness from below.

In particular, the operator L which is generated by the operation (6) is a self-adjoint operator. It is easy to verify that the equation

$$-u'' + (t^2 - \lambda)\, u = 0$$

with

$$\lambda = 2n + 1 \qquad (n = 0, 1, 2, \ldots)$$

is satisfied by the sequence of Tchebysheff-Hermite functions (cf. Section 11) which form a complete system in $L^2(-\infty, \infty)$. Therefore, the operator L has a pure point spectrum with the unique limit point at infinity. Thus, the inversion formulas connected with the operator L are indicated by the expansion in Tchebysheff-Hermite functions.

The example considered is particularly instructive. It shows that the property that the resolvent is a completely continuous operator, which, by Theorem 2 of Section 6 of Appendix II, always holds in the quasi-regular case, also can hold in other cases (even in case of minimal deficiency indices, as in the above example).

IV. The Bessel functions. Among the various differential operations which lead to Bessel functions the most important one has the form

$$l = -\frac{d^2}{dt^2} + \frac{\nu^2 - \frac{1}{4}}{t^2}. \tag{8}$$

We consider this operation with the parameter $\nu \geqq 0$. As far as the interval is concerned it is natural to study the following three cases:

(α)	$0 < t \leqq 1$	one singular end point,
(β)	$1 \leqq t < \infty$	one singular end point,
(γ)	$0 < t < \infty$	two singular end points.

The general integral of the equation

$$l\,[u] - \lambda u = 0 \tag{9}$$

with $\lambda \neq 0$ has the form

$$u(t;\lambda) = At^{\frac{1}{2}}J_\nu(t\sqrt{\lambda}) + Bt^{\frac{1}{2}}Y_\nu(t\sqrt{\lambda}). \tag{10}$$

Here A and B are arbitrary constants and $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind, respectively. They are defined by the formulas

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\ \Gamma(\nu+k+1)},$$

$$Y_\nu(z) = \frac{J_\nu(z)\cos \pi\nu - J_{-\nu}(z)}{\sin \pi\nu}.$$

We assume that $\Im \lambda > 0$ and $t > 0$. Then the value $z = t\sqrt{\lambda}$ satisfies the inequality

$$0 < \arg z < \frac{\pi}{2}.$$

For $|\arg z| < \pi$ and $|z| \to \infty$ the following asymptotic formulas hold:

$$J_\nu(z) + iY_\nu(z) = H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)},$$

$$J_\nu(z) - iY_\nu(z) = H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)}.$$

We substitute these formulas in (10) and find that equation (9) has not more than one solution which belongs to $L^2(1, \infty)$. But one such solution must exist. Hence, this solution is the function

$$u_1(t;\lambda) = AH_\nu^{(1)}(t\sqrt{\lambda}).$$

The deficiency indices of the operator L in case (β) are (1, 1).

We shall find the deficiency indices of the operator L in case α. Since, as $z \to 0$,

$$z^{\frac{1}{2}}J_\nu(z) \sim \frac{z^{\nu+\frac{1}{2}}}{2^\nu\, \Gamma(\nu+1)},$$

and, on the other hand, $z^{\frac{1}{2}}Y_\nu(z)$ does not belong to $L^2(0, 1)$ for $\nu \geqq 1$ and belongs to $L^2(0, 1)$ for $0 \leqq \nu < 1$; in the case α the deficiency indices of L are

$$(2, 2) \text{ for } 0 \leqq \nu < 1,$$

$$(1, 1) \text{ for } \nu \geqq 1.$$

Therefore, in case γ the deficiency indices of L are

$$\begin{aligned}&(1, 1) \text{ for } 0 \leqq \nu < 1,\\ &(0, 0) \text{ for } \nu \geqq 1.\end{aligned}$$

In each of the enumerated cases the differential operation (8) generates certain inversion formulas.

We obtain a unique pair of formulas for the interval $(0, \infty)$ if $\nu \geqq 1$. These formulas have the form

$$\left\{\begin{aligned} g(\lambda) &= \int_0^\infty \sqrt{\lambda t}\, J_\nu(\lambda t) f(t)\, dt,\\ f(t) &= \int_0^\infty \sqrt{\lambda t}\, J_\nu(\lambda t) g(\lambda)\, d\lambda. \end{aligned}\right. \tag{11}$$

We have here not only a unitary but a self-adjoint operator in $L^2(0, \infty)$. It is called the *Hankel transform*. To derive the inversion formulas (11) rather simply, we can apply one of the methods indicated for obtaining the Fourier-Plancherel inversion formulas. It turns out that formulas (11) hold also for $\nu \geqq 0$ but, for $0 \leqq \nu < 1$, are not the pair of unique inversion formulas on the semi-axis[17] generated by formula (8).

We consider the case of the interval $(0, 1]$ for $\nu \geqq 1$ and we assume that the boundary condition at the regular end point has the form

$$\varphi(1) = 0.$$

Hence, we can let

$$u(t;\lambda) = \frac{\pi}{2}\sqrt{t}\{J_\nu(t\sqrt{\lambda})\, Y_\nu(\sqrt{\lambda}) - Y_\nu(t\sqrt{\lambda}) J_\nu(\sqrt{\lambda})\}.$$

Furthermore, let

$$v(t;\lambda) = \frac{\pi}{2}\sqrt{t\lambda}\{J_\nu(t\sqrt{\lambda})\, Y_\nu'(\sqrt{\lambda}) - Y_\nu(t\sqrt{\lambda}) J_\nu'(\sqrt{\lambda})\}.$$

Then

$$K(t, s; z) = \begin{cases} u(t;z)\,[v(s;z) + m(z)\, u(s;z)] & (t \leqq s)\\ u(s;z)\,[v(t;z) + m(z)\, u(t;z)] & (t > s). \end{cases}$$

The function $m(z)$ is defined with the aid of the condition that the solution

$$v(t; z) + m(z)\; u(t; z)$$

---

[17] The formulas (11) are valid also for $\nu \geqq -\frac{1}{2}$ and also in a slightly different form for complex ν such that $\nu > -\frac{1}{2}$.

belongs to the space $L^2(0, 1)$. This means that the function

$$\frac{\pi}{2}\sqrt{t}\{J_\nu(t\sqrt{z})\,[\sqrt{z}\,Y'_\nu(\sqrt{z})+m(z)Y_\nu(\sqrt{z})]\}-$$

$$-\frac{\pi}{2}\sqrt{t}\{Y_\nu(t\sqrt{z})\,[\sqrt{z}J'_\nu(\sqrt{z})+m(z)J_\nu(\sqrt{z})\,]\}$$

should reduce to the first term. Thus,

$$m(z)=-\sqrt{z}\,\frac{J'_\nu(\sqrt{z})}{J_\nu(\sqrt{z})}\,.$$

This function can be represented in the form

$$m(z)=-2\sum_{n=1}^{\infty}\frac{\lambda_n}{z-\lambda_n}$$

where the λ_n are the nonzero roots of the function $J_\nu(\sqrt{z})$. It follows that $\sigma(\lambda)$ is a piecewise constant function and that the sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ forms the spectrum. The eigenfunctions are

$$u_n(t)=u(t;\lambda_n)=\frac{\pi}{2}\sqrt{t}\,Y_\nu(\sqrt{\lambda_n})J_\nu(t\sqrt{\lambda_n})=-\frac{\sqrt{t}J_\nu(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}J'_\nu(\sqrt{\lambda_n})}\,.$$

The inversion formulas reduce to the expansion formula

$$f(t)=\sum_{n=1}^{\infty}\frac{2\sqrt{t}J_\nu(t\sqrt{\lambda_n})}{J'^2_\nu(\sqrt{\lambda_n})}\int_0^1\sqrt{s}\,J_\nu(s\sqrt{\lambda_n})f(s)\,ds,$$

which is known as the *Fourier-Bessel series.*

If at the regular end point the general condition

$$\varphi'(1)=-\theta\,\varphi(1)$$

was assumed, then we would arrive at the so-called *Fourier-Dienes series.*

We do not consider the case of the interval (0, 1) for $0 \leqq \nu < 1$. We remark only that in this case the method that we used in connection with the Legendre operation can be adapted.

In addition, we leave to the reader the considerations of the case of the interval $(1, \infty)$ and the deficiency indices (1, 1). If we assume at the regular end point the condition

$$\varphi(1)=0,$$

then we find, by the method used repeatedly,

$$m(z)=-\sqrt{z}\,\frac{J'_\nu(\sqrt{z})+iY'_\nu(\sqrt{z})}{J_\nu(\sqrt{z})+iY_\nu(\sqrt{z})},$$

and we obtain the *Weber inversion formulas*

$$\Phi(\lambda) = \int_1^\infty \sqrt{t}\, \{J_\nu(t\sqrt{\lambda})\, Y_\nu(\sqrt{\lambda}) - Y_\nu(t\sqrt{\lambda})\, J_\nu(\sqrt{\lambda})\} f(t)\, dt,$$

$$f(t) = \int_0^\infty \sqrt{t}\, \frac{J_\nu(t\sqrt{\lambda})\, Y_\nu(\sqrt{\lambda}) - Y_\nu(t\sqrt{\lambda})\, J_\nu(\sqrt{\lambda})}{2\{J_\nu^2(\sqrt{\lambda}) + Y_\nu^2(\sqrt{\lambda})\}}\, \Phi(\lambda)\, d\lambda.$$

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Translator's Note: No bibliography is included in this volume. All references to bibliography in this book are to the complete bibliography of the German edition, which is reproduced in Volume I of this translation.

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